GEOMETRY OF CURVES AND SURFACES

Homework 4 (for the week Feb 12 - Feb 19)

(1) Let
$$\mathbf{x}\colon \mathbb{R}^2 \to \mathbb{R}^3,\ \ (u,v) \mapsto (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2).$$
Show that, at every point of $\mathbb{R}^2$, $\mathbf{x}_u \times \mathbf{x}_v \neq \bar{0}$. (Here $\bar{0}$ denotes the zero vector in $\mathbb{R}^3$.)

(2) For each of the following, check if $\mathbf{x}\colon \mathbb{R}^2 \to \mathbb{R}^3$ is injective and calculate $\mathbf{x}_u \times \mathbf{x}_v$.
  (a) $\mathbf{x}(u,v) = (u, v, uv)$
  (b) $\mathbf{x}(u,v) = (u, v^2, v^3)$
  (c) $\mathbf{x}(u,v) = (u + u^2, v, v^2)$

(3) The map
$$\mathbf{x}\colon U = \mathbb{R} \times (0, 2\pi) \to \mathbb{R}^3,\ \ (u,v) \mapsto (\frac{\cos v}{\cosh u}, \frac{\sin v}{\cosh u}, \tanh u),$$
is a coordinate patch of the unit sphere
$$\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$
Show that $\mathbf{x}$ is one-to-one and that $\mathbf{x}_u \times \mathbf{x}_v \neq \bar{0}$ at every point of U. Also show that $\mathbf{x}(U) \subset \mathbb{S}^2$.

(4) Find the equation of the tangent plane of the following coordinate patch at the point $\mathbf{x}(0,0)$. (You do not have to check that the patch is regular.)
$$\mathbf{x}(u,v) = ((2 + \sin v)\cos u, (2 + \sin v)\sin u, \cos v)$$

(5) Find the equation of the tangent plane of each of the following coordinate patches at the indicated points:
  (a) $\mathbf{x}(u,v) = (u, v, u^2 - v^2)$, $(1,1,0)$
  (b) $\mathbf{x}(r,\theta) = (r\cosh\theta, r\sinh\theta, r^2)$, $(1,0,1)$
  (You do not have to check that the patches are regular.)

Date: February 11, 2016.