GEOMETRY OF CURVES AND SURFACES

Homework 10 (for the week Apr 11 - Apr 15)

Let γ be a regular but not necessary unit-speed curve on a smooth surface S. Let k_g denote the geodesic curvature of γ. Let N denote the unit normal vector to S. Then

$$k_g = \frac{\gamma'' \cdot (N \times \gamma')}{\|\gamma'\|^3}$$

(1) Consider the surface S parametrized by

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, v).$$

Find the geodesic curvatures of the following curves on S:
(a) $\alpha(t) = (a \cos t, a \sin t, t) = \mathbf{x}(a, t)$, where $a \in \mathbb{R}$,
(b) $\beta(t) = (t \cos b, t \sin b, b) = \mathbf{x}(t, b)$, where $b \in \mathbb{R}$.

(2) Consider the cylinder $S\colon \mathbf{x}(u, v) = (\cos u, \sin u, v)$. Find the geodesic curvature of the following curve on S:

$$\alpha(t) = (\cos t, \sin t, a \cos t) = \mathbf{x}(t, a \cos t).$$

For which values of a is the curve a geodesic?

(3) Let the surface S be as in Exercise 2. Find the geodesic curvature of the following curve on S:

$$\beta(t) = (\cos \omega t, \sin \omega t, at + b) = \mathbf{x}(\omega t, at + b).$$

For which values of ω, a and b is the curve a geodesic?

(4) Let S be the sphere of radius R. Then

$$\mathbf{x}(u, v) = (R \cos u \cos v, R \sin u \cos v, R \sin v)$$

is a regular coordinate patch for S. Show that for $\mathbf{x}$, $E_u = 0 = G_u$. Write down the geodesic equations for a curve γ in the image of $\mathbf{x}$.

(5) Let $\mathbf{x}(u, v)$ be a smooth coordinate patch with first and second fundamental forms

$$E(du)^2 + 2F du dv + G(dv)^2, \quad \text{and} \quad L(du)^2 + 2M du dv + N(dv)^2.$$

Show that

$$\mathbf{x}_{uv} = \Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + MN_x,$$

where N_x is the unit normal vector of the patch,

$$\Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG - F^2)} \quad \text{and} \quad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG - F^2)}.$$

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