22. Geodesics on Surfaces of Revolution

Let $X(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$ be a parametrization of a surface of revolution S. Assume

$$f > 0 \quad \text{and} \quad \left(\frac{df}{du}\right)^2 + \left(\frac{dg}{du}\right)^2 = 1.$$

Earlier: The first fundamental form of X is

$$(du)^2 + (f(u))^2 (dv)^2.$$

Thus, for X, $E = 1$, $F = 0$ and $G = (f(u))^2$.

The geodesic equations are

1) $\frac{d}{dt}(Eu' + Fv') = \frac{1}{2}[E_u (u')^2 + 2F_u u'v' + G_u (v')^2]$

2) $\frac{d}{dt}(Fu' + Gv') = \frac{1}{2}[E_v (u')^2 + 2F_v u'v' + G_v (v')^2]$.

They become:

1') $u'' = \frac{1}{2} \cdot 2f(u)\frac{df}{du}(v')^2 = f(u)\frac{df}{du}(v')^2$

2') $\frac{d}{dt}((f(u))^2 v') = \frac{1}{2} \cdot 0 = 0.$

For a geodesic $X(u(t), v(t)) = \gamma(t)$,

$\gamma'(t) = u' \frac{dX}{du} + v'\frac{dX}{dv}$, and

$\gamma' \cdot \gamma' = E(u')^2 + 2Fu'v' + G(v')^2.$

If γ is a unit-speed geodesic on S, then

3') $(u')^2 + (f(u))^2 (v')^2 = 1.$ ← (This holds for unit-speed curves, not only for geodesics.)

We therefore obtain the following proposition:

Proposition 22.1. Let S be a surface of revolution parametrized by

$$X(u,v) = (f(u)\cos v,\ f(u)\sin v,\ g(u)),$$

where $f > 0$, and $\left(\frac{df}{du}\right)^2 + \left(\frac{dg}{du}\right)^2 = 1$. Then

1) Every meridian on S is a geodesic.

2) A parallel $u = u_0$ is a geodesic if and only if $\frac{df}{du} = 0$ when $u = u_0$

Recall: For a surface of revolution parametrized by $X(u,v) = (f(u)\cos v,\ f(u)\sin v,\ g(u))$,

1) the curve $\alpha(u) = (f(u), 0, g(u))$ is called the profile curve,

2) the circles obtained by rotating a fixed point on the profile curve about the z-axis, are called parallels,

3) the curve on the surface obtained by rotating the profile curve through a fixed angle is called a meridian.

proof of Prop. 22.1.

1) On a meridian v is constant. Then the equation $2'$ holds. Equation $3' \Rightarrow u' = \pm 1 \Rightarrow u'' = 0 \Rightarrow$ the equation $1'$ holds.

2) Assume $u = u_0$ is constant. Then $3' \Rightarrow v' = \pm \frac{1}{f(u)} \neq 0$. Thus $1'$ only holds if $\frac{df}{du} = 0$. Conversely, assume $\frac{df}{du} = 0$ when $u = u_0$. Then $1'$ holds. Since $v' = \pm \frac{1}{f(u_0)}$ and $f(u) = f(u_0)$ are constant, it follows that $2'$ holds. $\square$