23. Gauss' Theorema Egregium

Theorem 23.1. (Gauss' Theorema Egregium)

Let S_1 and S_2 be smooth surfaces, and let $\phi: S_1 \to S_2$ be a local isometry. Let K_1 and K_2 denote the Gaussian curvatures of S_1 and S_2, respectively. Then $K_1(p) = K_2(\phi(p))$, for all $p \in S_1$.

To be able to prove Theorem 23.1, we first need some technical results. This is the first:

Prop. 23.2. (Gauss equations) Let $X(u,v)$ be a regular surface patch with 1st and 2nd fundamental forms

$$E(du)^2 + 2F\,du\,dv + G(dv)^2 \quad \text{and}$$

$$L(du)^2 + 2M\,du\,dv + N(dv)^2.$$

Then

$$X_{uu} = \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + L N_X,$$

$$X_{uv} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + M N_X$$

$$X_{vv} = \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + N N_X,$$

where $N_X = \dfrac{X_u \times X_v}{\|X_u \times X_v\|}$ is a unit normal to X, and

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG-F^2)} \qquad \Gamma^2_{11} = \frac{2EF_u - EE_v - FE_u}{2(EG-F^2)}$$

$$\Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG-F^2)} \qquad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG-F^2)}$$

$$\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG-F^2)} \qquad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG-F^2)}$$

The coefficients Γ are called Christoffel symbols. They only depend on E, F and G.

proof. Notice that $\{x_u, x_v, N_x\}$ forms a basis of $\mathbb{R}^3$. Thus there are real valued functions, $\alpha_i, \beta_i, \gamma_i$, $1 \leq i \leq 3$, s.t.

$$(*) \quad \begin{cases} x_{uu} = \alpha_1 x_u + \alpha_2 x_v + \alpha_3 N_x, \\ x_{uv} = \beta_1 x_u + \beta_2 x_v + \beta_3 N_x, \\ x_{vv} = \gamma_1 x_u + \gamma_2 x_v + \gamma_3 N_x. \end{cases}$$

Take the dot product of each eqn with N_x:
$\Rightarrow$

$$L = x_{uu} \cdot N_x = \alpha_3$$

$$M = x_{uv} \cdot N_x = \beta_3$$

$$N = x_{vv} \cdot N_x = \gamma_3.$$

Next, take the dot product of the eqns in $(*)$ with x_u and x_v. $\Rightarrow$ 6 equations that determine $\alpha_i, \beta_i, \gamma_i$, $1 \leq i \leq 2$. For example, take the dot product of the 1st eqn with x_u and x_v.
$\Rightarrow$

$$\underbrace{\alpha_1 x_u \cdot x_u + \alpha_2 x_v \cdot x_u}_{E\alpha_1 + F\alpha_2} = x_{uu} \cdot x_u = \frac{1}{2} E_u$$

$$\underbrace{\alpha_1 x_u \cdot x_v + \alpha_2 x_v \cdot x_v}_{F\alpha_1 + G\alpha_2} = x_{uu} \cdot x_v = (x_u \cdot x_v)_u - x_u \cdot x_{uv}$$
$$= F_u - \frac{1}{2} E_v$$

Then
$$E\alpha_1 + F\alpha_2 = \frac{1}{2} E_u \qquad | \cdot F$$

$$F\alpha_1 + G\alpha_2 = F_u - \frac{1}{2} E_v \qquad | \cdot E$$

$$\Rightarrow \quad FE\alpha_1 + F^2\alpha_2 = \tfrac{1}{2}FE_u$$

$$\underline{-FE\alpha_1 - GE\alpha_2 = \tfrac{1}{2}EE_v - EF_u}$$

$$(F^2 - GE)\alpha_2 = \tfrac{1}{2}(FE_u - 2EF_u + EE_v)$$

$$\Rightarrow \quad \alpha_2 = \frac{2EF_u - EE_v - FE_u}{2(GE - F^2)} = \Gamma^2_{11}$$

Also

$$EG\alpha_1 + FG\alpha_2 = \tfrac{1}{2}GE_u$$

$$\underline{-F^2\alpha_1 - FG\alpha_2 = \tfrac{1}{2}FE_v - FF_u}$$

$$(EG - F^2)\alpha_1 = \tfrac{1}{2}(GE_u + FE_v - 2FF_u)$$

$$\Rightarrow \quad \alpha_1 = \frac{GE_u + FE_v - 2FF_u}{2(EG - F^2)} = \Gamma^1_{11}.$$

Similarly, one can solve the other coefficients in $(*)$. $\square$

<u>Proposition 23.3.</u> (Codazzi–Mainardi Equations)

Let $E(du)^2 + 2F\,du\,dv + G(dv)^2$ and $L(du)^2 + 2M\,du\,dv + N(dv)^2$

be the 1^{st} and 2^{nd} fundamental forms of a surface patch $X(u,v)$. Define the Christoffel symbols as in Prop. 23.2. Then

$$L_v - M_u = L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11},$$

$$M_v - N_u = L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12}.$$

Proposition 23.4. (Gauss Equations)

Let K be the Gaussian curvature of the surface patch $X(u,v)$ in Prop. 23.3. Then

$$EK = (\Gamma_{11}^2)_v - (\Gamma_{12}^2)_u + \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{12}^1\Gamma_{11}^2 - (\Gamma_{12}^2)^2$$

$$FK = (\Gamma_{12}^1)_u - (\Gamma_{11}^1)_v + \Gamma_{12}^2\Gamma_{12}^1 - \Gamma_{11}^2\Gamma_{22}^1$$

$$FK = (\Gamma_{12}^2)_v - (\Gamma_{22}^2)_u + \Gamma_{12}^1\Gamma_{12}^2 - \Gamma_{22}^1\Gamma_{11}^2$$

$$GK = (\Gamma_{22}^1)_u - (\Gamma_{12}^1)_v + \Gamma_{22}^1\Gamma_{11}^1 + \Gamma_{22}^2\Gamma_{12}^1 - (\Gamma_{12}^1)^2 - \Gamma_{12}^2\Gamma_{22}^1.$$

proof of Prop. 23.3 and Prop 23.4:

Since X is smooth, it follows that $(X_{uu})_v = (X_{uv})_u$. By Prop 23.2,

$$(\Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + LN_X)_v = (\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + MN_X)_u .$$

$$\Rightarrow \frac{\partial\Gamma_{11}^1}{\partial v} X_u + \Gamma_{11}^1 X_{uv} + \frac{\partial\Gamma_{11}^2}{\partial v} X_v + \Gamma_{11}^2 X_{vv} + L_v N_X + L(N_X)_v$$

$$= \frac{\partial\Gamma_{12}^1}{\partial u} X_u + \Gamma_{12}^1 X_{uu} + \frac{\partial\Gamma_{12}^2}{\partial u} X_v + \Gamma_{12}^2 X_{vu} + M_u N_X + M(N_X)_u$$

$$\Rightarrow \left(\frac{\partial\Gamma_{11}^1}{\partial v} - \frac{\partial\Gamma_{12}^1}{\partial u}\right) X_u + \left(\frac{\partial\Gamma_{11}^2}{\partial v} - \frac{\partial\Gamma_{12}^2}{\partial u}\right) X_v + (L_v - M_u) N_X$$

$$= \Gamma_{12}^1 X_{uu} + (\Gamma_{12}^2 - \Gamma_{11}^1) X_{uv} - \Gamma_{11}^2 X_{vv} - L(N_X)_v + M(N_X)_u .$$

$$= \Gamma_{12}^1 (\underbrace{\Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + LN_X}_{X_{uu}}) + (\Gamma_{12}^2 - \Gamma_{11}^1)(\underbrace{\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + MN_X}_{X_{uv}})$$

$$- \Gamma_{11}^2 (\underbrace{\Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + NN_X}_{X_{vv}}) - L(N_X)_v + M(N_X)_u . \qquad (*)$$

Here $N_X \cdot N_X = 1 \Rightarrow N_X \cdot (N_X)_u = 0 = N_X \cdot (N_X)_v .$

$\Rightarrow (N_x)_u$ and $(N_x)_v$ are linear combinations of X_u and X_v.

Collect the N_x-components of each side of the equation:

$$\Rightarrow \quad L_v - M_u = L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11}.$$

This is the first of the Codazzi-Mainardi equations.

Similarly: $(X_{uv})_v = (X_{vv})_u$, collect the N_x-components

$\Rightarrow$ the 2^{nd} C-M equation.

Collect the X_u-components in $(*)$:

As in the proof of Prop. 17.8, write

$$(N_x)_u = aX_u + bX_v, \quad (N_x)_v = cX_u + dX_v,$$

where $\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -g^{-1}h$. Then

$$(\Gamma^1_{11})_v - (\Gamma^1_{12})_u = \Gamma^1_{12}\Gamma^1_{11} + (\Gamma^2_{12} - \Gamma^1_{11})\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22}$$

$$+ (-Lc \qquad + Ma) \qquad (**)$$

The proof of Prop. 17.8:

$$\begin{aligned}
-L &= aE + bF \quad | \cdot G \\
-M &= cE + dF \quad | \cdot G \\
-N &= cF + dG \quad | \cdot (-F) \\
-M &= aF + bG \quad | \cdot (-F)
\end{aligned}$$

$$\begin{aligned}
-LG &= aEG + bFG \\
MF &= -aF^2 - bFG \\
\hline
MF - LG &= a(EG - F^2) \quad \Rightarrow \underline{a = \frac{MF - LG}{EG - F^2}}
\end{aligned}$$

$$\begin{aligned}
-MG &= cEG + dFG \\
FN &= -cF^2 - dFG \\
\hline
FN - MG &= c(EG - F^2) \quad \Rightarrow \underline{c = \frac{FN - MG}{EG - F^2}}
\end{aligned}$$

Therefore,

$$-Lc + Ma = -\frac{LFN - LMG - M^2F + MLG}{EG - F^2} = -\frac{F(LN - M^2)}{EG - F^2} = -FK$$

Substitute in $(**)$ $\Rightarrow$

$$(\Gamma^1_{11})_v - (\Gamma^1_{12})_u = \Gamma^1_{12}\Gamma^1_{11} + (\Gamma^2_{12} - \Gamma^1_{11})\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} - FK = \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} - FK$$

$$\Rightarrow \quad FK = (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22}$$

This is the second of the Gauss equations.

Similarly, one can prove the remaining three of the Gauss equations:

Collect: 1) the coefficients of X_u in $(X_{uu})_v = (X_{uv})_u$

2) the coefficients of X_u in $(X_{uv})_v = (X_{vv})_u$

3) the coefficients of X_v in $(X_{uv})_v = (X_{vv})_u$.

$\square$

Let's return to Gauss' Theorema Egregium (Theorem 23.1) which says that the Gaussian curvature of a surface is preserved by local isometries. It is possible to find an expression for K in terms of E, F and G by substituting the formulas for the Christoffel symbols in the equations of 23.4. Doing that would give four formulas, one from each equation, the formulas would then turn out to be the same. The result is the following:

Corollary 23.5. The Gaussian curvature is

$$K = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG - F^2)^2}$$

proof. Skip.

Instead of proving the general case, we prove a couple of special cases:

Corollary 23.6.

1) If $F = 0$, then

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right).$$

2) If $E = 1$ and $F = 0$, then

$$K = \frac{-1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2}.$$

proof.

1) Assume $F = 0$. Then, by Prop. 23.2,

$$\Gamma^1_{11} = \frac{E_u}{2E}, \quad \Gamma^2_{11} = -\frac{E_v}{2G}, \quad \Gamma^1_{12} = \frac{E_v}{2E}$$

$$\Gamma^2_{12} = \frac{G_u}{2G}, \quad \Gamma^1_{22} = -\frac{G_u}{2E}, \quad \Gamma^2_{22} = \frac{G_v}{2G}.$$

Substitute these in the first of the Gauss equations (Prop. 23.4.) gives

$$EK = (\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})^2$$

$$= -\left(\frac{E_v}{2G}\right)_v - \left(\frac{G_u}{2G}\right)_u + \frac{E_u}{2E}\frac{G_u}{2G} - \frac{E_v}{2G}\frac{G_v}{2G} + \frac{E_v}{2E}\frac{E_v}{2G} - \left(\frac{G_u}{2G}\right)^2 \quad \Big| \cdot \left(-2\sqrt{\tfrac{G}{E}}\right)$$

$$\Rightarrow -2K\sqrt{EG} = 2\sqrt{\frac{G}{E}}\left(\frac{2GE_{vv} - 2E_vG_v}{4G^2} + \frac{2GG_{uu} - 2(G_u)^2}{4G^2}\right)$$

$$- \frac{E_uG_u}{2E^{3/2}\sqrt{G}} + \frac{E_vG_v}{2G^{3/2}\sqrt{E}} - \frac{(E_v)^2}{2E^{3/2}\sqrt{G}} + 2\sqrt{\frac{G}{E}}\left(\frac{G_u}{2G}\right)^2$$

$$= \frac{E_{vv} + G_{uu}}{(EG)^{1/2}} + \underbrace{\sqrt{\frac{G}{E}} \cdot \frac{-E_vG_v}{G^2} + \frac{E_vG_v}{2G^{3/2}\sqrt{E}}}_{\frac{-E_vG_v}{2G^{3/2}E^{1/2}}} - \frac{(E_v)^2}{2E^{3/2}G^{1/2}}$$

$$- \frac{(G_u)^2}{G^{3/2}E^{1/2}} - \frac{E_uG_u}{2E^{3/2}G^{1/2}} + \frac{(G_u)^2}{2G^{3/2}E^{1/2}}$$

$$= \frac{E_{vv} + G_{uu}}{(EG)^{1/2}} - \frac{E_v(EG_v + GE_v)}{2(EG)^{3/2}} - \frac{G_u(EG_u + GE_u)}{2(EG)^{3/2}}$$

$$= \frac{E_{vv}}{(EG)^{1/2}} - \frac{1}{2}\frac{E_v(EG)_v}{(EG)^{3/2}} - \frac{1}{2}\frac{G_u(EG)_u}{(EG)^{3/2}} + \frac{G_{uu}}{(EG)^{1/2}}$$

$$= \left(\frac{E_v}{(EG)^{1/2}}\right)_v + \left(\frac{G_u}{(EG)^{1/2}}\right)_u$$

∴ Equation 1

2) Assume then that $F = 0$ and $E = 1$. Then $E_v = 0$.
Then $1 \Rightarrow$

$$-2K\sqrt{G} = \left(\frac{G_u}{(\underset{=1}{E}G)^{1/2}}\right)_u$$

$$\Rightarrow \quad K = \frac{-1}{2\sqrt{G}}\frac{\partial}{\partial u}\Big(\underbrace{\frac{G_u}{\sqrt{G}}}_{2\cdot\frac{\partial}{\partial u}\sqrt{G}}\Big) = \frac{-1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2}.$$

∴ Equation 2

□

<u>Example</u> Let $X(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$ parametrize a surface of revolution. Assume $f > 0$ and $(f')^2 + (g')^2 = 1$. Earlier we saw that $E = 1$, $F = 0$ and $G = (f(u))^2$. Thus we can apply Corollary 23.6 to calculate the Gaussian curvature of the surface.

$$G = (f(u))^2 \Rightarrow \frac{\partial\sqrt{G}}{\partial u} = f'(u)$$

$$\Rightarrow \frac{\partial^2\sqrt{G}}{\partial u^2} = f''(u)$$

Thus
$$K = \frac{-1}{\sqrt{G}} \frac{\partial^2\sqrt{G}}{\partial u^2} = \frac{-1}{f(u)} f''(u) = -\frac{f''}{f}.$$