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Author(s): Richard M. Friedberg
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THREE THEOREMS ON RECURSIVE ENUMERATION. I. DECOMPOSITION. II. MAXIMAL SET. III. ENUMERATION WITHOUT DUPLICATION

RICHARD M. FRIEDBERG

In this paper we shall prove three theorems about recursively enumerable sets. The first two answer questions posed by Myhill [1].

The three proofs are independent and can be presented in any order. Certain notations will be common to all three. We shall denote by "R_e" the set enumerated by the procedure of which e is the Gödel number. In describing the construction for each proof, we shall suppose that a clerk is carrying out the simultaneous enumeration of $R_0, R_1, R_2, \ldots,$ in such a way that at each step only a finite number of sets have begun to be enumerated and only a finite number of the members of any set have been listed. (One plan the clerk can follow is to turn his attention at Step a to the enumeration of R_e where $e+1$ is the number of prime factors of a. Then each R_e receives his attention infinitely often.) We shall denote by "R_e^a" the set of numbers which, at or before Step a, the clerk has listed as members of R_e. Obviously, all the R_e^a are finite sets, recursive uniformly in e and a. For any a we can determine effectively the highest e for which R_e^a is not empty, and for any a, e we can effectively find the highest member of R_e^a, just by scanning what the clerk has done by Step a. Additional notations will be introduced in the proofs to which they pertain.

Theorem 1. *Every nonrecursive recursively enumerable set is the union of two disjoint nonrecursive recursively enumerable sets.*

Proof. Let us be given the Gödel number u of a recursively enumerable set R_u. We shall show how to enumerate two disjoint sets P and Q whose union is R_u. Then we shall prove that neither P nor Q is recursive if R_u is not recursive.

The enumeration of P and Q will be carried out *pari passu* with the clerk's enumeration of the R_e's. "P^a" or "Q^a" will denote the set of numbers which, at or before Step a, have been made members of P or Q, respectively. A number e will be called *satisfied* at Step a if R_e^a intersects both P^a and Q^a. (If e ever is satisfied, R_e cannot be the complement of either P or Q.)

Every time the clerk lists a new member of R_u, we shall put that number either into P or into Q. Suppose the number is n, and it is listed at Step a; that is, $n \in R_u^a - R_u^{a-1}$. If every e such that $n \in R_e^a$ is already satisfied, then we put n into P. Otherwise, we *attack* the lowest unsatisfied e such that $n \in R_e^a$. If R_e^a, for the e under attack, intersects neither P^{a-1} nor Q^{a-1},

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then we put n into P, so that R_e^a intersects P^a. Otherwise we put n into P or Q according as $R_e^a \cap P^{a-1}$ or $R_e^a \cap Q^{a-1}$ is empty, so that e now becomes satisfied.

Now, neither P nor Q is recursive if R_u is not recursive. For if either one of P or Q is recursive, its complement is some recursively enumerable set R_c. This implies that

(1) R_c includes the complement of R_u, and

(2) c never becomes satisfied.

Note that no number is attacked more than twice, for after two attacks it is satisfied, and only unsatisfied numbers are attacked. Therefore there is a step a_0 in our construction after which no number $\leq c$ is ever attacked. Then after Step a_0, the clerk never lists a member of R_u that has previously been listed as a member of R_c. For if he did, c being unsatisfied by (2), we should attack either c or some $e < c$.

This argument gives us a way of enumerating the complement of R_u. Simply write down every number that the clerk lists as a member of R_c, *provided* that it was not previously listed as a member of R_u *and* that it still has not been listed as a member of R_u by Step a_0. The resulting set contains all the non-members of R_u, by (1); and it contains no members of R_u, by the argument of the preceding paragraph. Since its complement is recursively enumerable, R_u is recursive. This proves the theorem.

THEOREM 2. *There exists a recursively enumerable set M (a "maximal" set) such that*

(3) *$\bar{M}$ is infinite, and*

(4) *there is no recursively enumerable set R such that $R \cap \bar{M}$ and $\bar{R} \cap \bar{M}$ are both infinite.*

PROOF. As we construct M, we shall use the notation "M^a", to denote the set of numbers which we have put into M at or before Step a. At Step a, for any particular e, n, we say that n inhabits the i-th e-state if $n \notin M^a$ and $i = \Sigma_{n \in R_{e'}^a, e' \leqq e} 2^{e-e'}$ where the summation is made over all $e' \leqq e$ such that $n \in R_{e'}^a$. The i-th e-state is *lower* than the j-th if $i < j$. It is evident that at the beginning of the construction every number inhabits the 0-th e-state. As the construction proceeds, a number n may sometimes move from a lower to a higher, but never from a higher to a lower e-state. If n is eventually put into M, n ceases then to inhabit any e-state. If $n \in \bar{M}$, n will always inhabit some e-state, and since there are only 2^{e+1} e-states, n must eventually inhabit some e-state which it continues to inhabit forever. n is then called a *resident* of that e-state. An e-state is *well-resided* if it has an infinite number of residents. An e-state is *well-habited* if each of an infinite set of numbers inhabits it at one step or another.

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M is constructed as follows. When, at Step a, the clerk lists any number n, not in M^{a-1}, as a new member of any set R_{e_0}, we examine every $e \geqq e_0$ such that $n \in R_e{}^a$. There are only finitely many such e. For each e examined, if more than e numbers $< n$ inhabit the e-state which is one lower than that of n, we put all but the e lowest of those numbers into M.

An example may prevent misunderstanding. Suppose the clerk lists 21 as a new member of R_2. Suppose that 21 has previously been listed as a member of R_0, of R_3, and of R_5, but not of M, nor of any other R_e. Then 21 is now in the 1-st 0-state, in the 2-nd 1-state, in the 5-th 2-state, in the 11-th 3-state, in the 22-nd 4-state, and in the 45-th 5-state. The values of e to be examined are 2, 3, and 5. Suppose that, of the numbers less than 21, only 6 is in the 4-th 2-state; 1, 9, 11, and 13 are in the 10-th 3-state; and 4, 8, 10, 16, 18, 19, and 20 are in the 44-th 5-state. Then 13, 19, and 20 are the numbers we must put into M.

M shall contain only those numbers which are put into it according to the foregoing prescription. To show that M is maximal, we prove two lemmas.

LEMMA 1. *For each c, the highest well-habited c-state has at least c residents.*

PROOF. The 0-th c-state is well-habited. Hence there exists a highest well-habited c-state, say the i_m-th. Then only finitely many numbers are fated ever to inhabit a c-state higher than the i_m-th. Let a_0 be a step in the construction at which each of these numbers has either entered the c-state in which it is to reside or been put into M, so that after Step a_0 no number enters a c-state higher than the i_m-th. Since the i_m-th c-state is well-habited and only a finite number of its inhabitants can have been put into M by Step a_0, there are infinitely many numbers which, at some step subsequent to a_0, will be found inhabiting the i_m-th c-state. Let n_0 be one of the c lowest of these numbers. By hypothesis, n_0 will never leave the i_m-th c-state for a higher one. Therefore n_0 is a resident of the i_m-th c-state, unless $n_0 \in M$. But the act of putting n_0 into M requires that, at some step $a > a_0$, a number $n > n_0$ is listed as a member of some R_{e_0}, and for some $e \geqq e_0$ there are at least e numbers $< n_0$ in the same e-state as n_0, and n is then in the next higher e-state. This is impossible. For if $e \geqq c$, all numbers in the same e-state as n_0 are also in the same c-state, and not more than $c-1$ of these are less than n_0. And if $e < c$, then $e_0 < c$, so that n, by being newly listed in R_{e_0}, has entered a new c-state, and this must be higher than the i_m-th c-state if n is in a higher e-state than n_0. This, by hypothesis, cannot occur after Step a_0. Therefore n_0 cannot be put into M and must be a resident of the i_m-th c-state. Since n_0 could have been any of c numbers, the i_m-th c-state has at least c residents.

LEMMA 2. *For any c, not more than one c-state is well-resided.*

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PROOF. Let the highest well-habited c-state be the i_m-th. Then higher c-states, not being well-habited, cannot be well-resided. On the other hand, if $i < i_m$, the i-th c-state has no more than c residents. For supposing it has more, let n_0 be the $(c+1)$-th lowest resident of the i-th c-state. Let Step a_0 be a step at which this c-state is already inhabited by its $c+1$ lowest residents. Only finitely many numbers, at Step a_0, are in c-states higher than the i-th. But infinitely many numbers are fated eventually to inhabit the i_m-th c-state. Therefore infinitely many numbers must, after Step a_0, pass from a c-state $\leqq$ the i-th to one $>$ the i-th. Let the first number $> n_0$ that does so be called n. n can make this transition only by being listed, at some step a, as a new member of some R_{e_0} with $e_0 \leqq c$. n is not in M^{a-1}, or it could not be in any c-state. Consider the lowest e such that $R_e{}^a$ contains n but not n_0. This e must exist and be $\leq c$, since n is now in a higher c-state than n_0. For the same reason, no $R_{e'}{}^a$, where $e' < e$, contains n_0 but not n. Therefore the e-state of n_0 is one lower than that of n. Moreover, since by hypothesis n was not in a c-state higher than that of n_0 *before* n was listed as a member of $R_{e_0}{}^a$, e_0 must be $\leqq e$. Finally, the c lowest residents of the i-th c-state are all lower than n_0 and are all in the same e-state as n_0, since $e \leqq c$. Therefore all the conditions are satisfied for putting n_0 into M. This contradicts the hypothesis that n_0 is a resident of the i-th c-state. Therefore the i-th c-state does not, in fact, have more than c residents and is consequently not well-resided. This proves the lemma.

Since every resident is a member of $\bar{M}$, Lemma 1 implies that $\bar{M}$ has at least c members, for arbitrary c. Hence $\bar{M}$ is infinite and (3) is satisfied. For any R_e, all the members of $R_e \cap \bar{M}$ are residents of odd-numbered e-states, and all the members of $\bar{R}_e \cap \bar{M}$ are residents of even-numbered e-states. Therefore Lemma 2 implies (4). The theorem is proved.[1]

THEOREM 3. *There exists a sequence of $S_0, S_1, S_2, \ldots$ of uniformly recursively enumerable sets in which every recursively enumerable set occurs once and only once.*

PROOF. We shall give a procedure for listing simultaneously the members of $S_0, S_1, S_2, \ldots$ just as the clerk is listing the members of $R_0, R_1, R_2, \ldots, S_x{}^a$ shall be the set of numbers listed in S_x at or before Step a.

During the construction we shall establish certain conceptual relationships between values of e and values of x. Under certain circumstances we shall

[1] It can easily be proved that any maximal set is hyper-hyper-simple. Hence our construction of M answers Post's question ([2], p. 313) as to the existence of such sets. Namely, assume that M is *not* hyper-hyper-simple. Then, by definition, there exists an infinite recursively enumerable set $\{a_1, a_2, \ldots\}$ such that the sets R_{a_1}, $R_{a_2}, \ldots$ are all finite and mutually exclusive, and such that each contains some member of $\bar{M}$. Let $S = R_{a_2} \cup R_{a_4} \cup R_{a_6} \cup \ldots$. Then $S \cup \bar{M}$ and $\bar{S} \cup \bar{M}$ are both infinite, which is impossible. Q.E.D.

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call a certain x the *follower* of a certain e. This means that we intend, if convenient, to make S_x identical to R_e. At times we shall *release* an x; that is, we shall make it cease to be the follower of some e. This x will thereafter be *free*, and it will never again be the follower of any e. However, e, having lost its follower, may acquire a new one later. If some x, having been made a follower of some e, remains its follower forever, never being freed, then x will be called a *loyal* follower of e. If x is eventually freed, it will be called a *disloyal* follower. At any step the values of x which have not yet been followers of any e will be called *unused*. 0 shall always be unused, and S_0 shall be the empty set.

Each step a shall be dedicated to the *pursuit* of a certain e, according to some plan which causes every e to be pursued an infinite number of times during the construction. (For example, we may use the plan suggested above for the clerk, in which e is chosen so that $e+1$ is the number of prime factors of a.)

At any step a, let e_a be the value of e which is being pursued. One of three cases arises.

CASE 1: e_a has a follower, x, and there is some $e < e_a$ such that $R_e{}^a \cap L(x)$ is identical to $R_{e_a}{}^a \cap L(x)$, where $L(x)$ is the set of all integers less than x.

Then we release x.

CASE 2: Case 1 does not hold, and there exists an x such that $R_{e_a}{}^a$ is identical to $S_x{}^{a-1}$. Furthermore, either this x is the follower of some $e \leqq e_a$, or $x = 0$, or else x is free; if x is free, either $x \leqq e_a$ or x has previously been displaced by e_a. (See Case 3, Act C.)

Then we do nothing.

CASE 3: neither Case 1 nor Case 2 holds.

Then we perform four acts, some of which may be vacuous.

A. If e_a has no follower, we make the lowest unused x, other than 0, the follower of e_a.

B. We put into S_x, where x is the follower of e_a, all the members of $R_{e_a}{}^a$. (This makes $S_x{}^a$ identical to $R_{e_a}{}^a$, for there is no way in which S_x could previously have acquired any members that were not yet in R_{e_a}, either while x was unused or while x was the follower of e_a.)

C. If, for some $x' \neq x$, $S_{x'}{}^{a-1}$ is identical to $R_{e_a}{}^a$, then we put into $S_{x'}$ the lowest number not yet listed as a member of any R or S. This makes $S_{x'}{}^a$ different from $S_x{}^a$ and from all the other S's. When we perform this act, we say that x' is being *displaced* by e_a.

D. If the x' of Act C is the follower of some e', we release it.

LEMMA 3. *Let $\bar{e}$ be the smallest Gödel number of $R_{\bar{e}}$; i.e., let $R_{\bar{e}}$ differ from R_e for every $e < \bar{e}$. Then there exists an x such that S_x is identical to $R_{\bar{e}}$.*

PROOF. $\bar{e}$ cannot have an infinite number of disloyal followers. For

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eventually, for each $e < \bar{e}$, one of the two sets R_e and $R_{\bar{e}}$ must acquire a member n never to be acquired by the other. Thereafter, $R_e{}^a$ and $R_{\bar{e}}{}^a$ can never be identical, and $R_e{}^a \cap L(x)$ cannot be identical to $R_{\bar{e}}{}^a \cap L(x)$ for any $x > n$. When such an n has appeared for every $e < \bar{e}$, $\bar{e}$ can no longer lose any follower through Act D with $e_a < e$, and if Case 1 continues to occur with $e_a = \bar{e}$, $\bar{e}$ must eventually be found with a follower x greater than all the n's, so that Case 1 is impossible. Subsequently, any follower that $\bar{e}$ has will be loyal.

Let a_0 be a step after which $\bar{e}$ is destined never to lose a follower. Suppose Case 3 occurs infinitely often with $e_a = \bar{e}$. Then the first time it occurs after Step a_0, $\bar{e}$ acquires a follower through Act A, or else it already has one. In either case, this follower will be loyal. Call it x. S_x will be identical to $R_{\bar{e}}$, for no number will be made a member of S_x unless the clerk has already listed it in $R_{\bar{e}}$, and every number that the clerk lists in $R_{\bar{e}}$ will be put into S_x at the next occurrence of Case 3 with $e_a = \bar{e}$.

Suppose, on the other hand, that Case 3 occurs only finitely often with $e_a = \bar{e}$. Since Case 1 does not arise after Step a_0, Case 2 must occur infinitely often with $e_a = \bar{e}$. Each time Case 2 occurs, there must be an x such that $S_x{}^{a-1}$ is identical to $R_{\bar{e}}{}^a$. This x must either be a follower of some $e \leqq \bar{e}$, or be itself $\leq \bar{e}$, or have been displaced by $\bar{e}$ at a previous step. We shall show that x can take only a finite number of different values in occurrences of Case 2 with $e_a = \bar{e}$.

If $e < \bar{e}$, eventually either $R_{\bar{e}}$ has acquired a member that will never be acquired by R_e, or R_e has acquired a member that will never be acquired by $R_{\bar{e}}$. In the first case, $S_x{}^{a-1}$ can never subsequently be identical to $R_{\bar{e}}{}^a$ when x is a follower of e, for $S_x{}^{a-1}$ will have no members not in $R_e{}^{a-1}$. In the second case, if e subsequently acquires a new follower x, S_x will immediately acquire all the members listed already in R_e, and $S_x{}^{a-1}$ cannot thereafter be identical to $R_{\bar{e}}{}^a$. In either case, only a finite number of different followers of e can serve as the x in Case 2 with $e_a = \bar{e}$.

If $e = \bar{e}$, e has only a finite number of different followers during the construction, as we have already shown.

Only a finite number of values of x are $\leqq \bar{e}$.

Since, by hypothesis, Case 3 arises only finitely often with $e_a = \bar{e}$, only a finite number of x's are ever displaced by $\bar{e}$.

Therefore only a finite number of different numbers can serve as the x in Case 2 with $e_a = \bar{e}$. But Case 2 arises infinitely often. Therefore there is a single x such that $S_x{}^{a-1}$ is identical to $R_e{}^a$ on an infinite number of occasions. Then S_x must be identical to $R_{\bar{e}}$, for otherwise one of the two sets would eventually acquire a member never to be acquired by the other, and then $S_x{}^{a-1}$ could never again be identical to $R_{\bar{e}}{}^a$. This proves Lemma 3.

COROLLARY TO LEMMA 3. *0 is the only value of x that remains forever unused.*

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For whenever a new x becomes used, the lowest unused $x > 0$ is selected. Therefore, if any $x_0 > 0$ were forever unused, then all $x \geqq x_0$ would remain forever unused. But if x is forever unused, then S_x is empty. Thus any R_e that differed from S_x for all $x < x_0$ would differ from all S_x, contrary to Lemma 3.

LEMMA 4. *If $x \neq x'$, then S_x and $S_{x'}$ are not the same finite set.*

PROOF. If S_x is finite, then $S_x{}^a$ is identical to S_x for sufficiently high a. Similarly for $S_{x'}$, $S_{x'}{}^a$. But $S_x{}^a$ is never identical to $S_{x'}{}^a$ if $x \neq x'$, except while both x and x' are unused. The reader may verify this by examining Case 3; it is contrived so that all the $S_x{}^a$ for different used x's are different, provided that this was true of $a-1$. By the corollary to Lemma 3, x and x' cannot both remain unused indefinitely. Therefore, for all sufficiently high a, $S_x{}^a$ differs from $S_{x'}{}^a$, and hence S_x differs from $S_{x'}$.

LEMMA 5. *If $x \neq x'$, then S_x and $S_{x'}$ are not the same infinite set.*

PROOF. If $x = 0$, S_x is empty, not infinite. If $x > 0$, then x is eventually a follower of some e, by the corollary to Lemma 3. If x is disloyal, then after x is released S_x can acquire a new member only when x is displaced. x can be displaced only once by each $e < x$ and never by any $e \geqq x$. Therefore S_x is finite if x is disloyal. Hence, if S_x is infinite, x must be a loyal follower of some e. Similarly, if $S_{x'}$ is infinite, x' is a loyal follower of some e'. A single e cannot have more than one loyal follower, since it cannot have more than one follower at a time. Hence, if $x \neq x'$, then $e \neq e'$. We may suppose arbitrarily that $e < e'$. Case 3 must arise infinitely often with $e_a = e$; otherwise S_x could not be infinite. Therefore every number listed in R_e is subsequently listed in S_x, and so S_x is identical to R_e. Similarly, $S_{x'}$ is identical to $R_{e'}$. Therefore, if S_x and $S_{x'}$ are identical, R_e and $R_{e'}$ must be identical. But then $R_e{}^a \cap L(x')$ must be identical to $R_{e'}{}^a \cap L(x')$ for all sufficiently high a. Once this is true, Case 1 will arise the next time that $e_a = e'$, and x' will be released. This contradicts the assumption that x' is a loyal follower of e'. Therefore S_x and $S_{x'}$ cannot be the same infinite set.

Every recursively enumerable set R occurs in the sequence $S_0, S_1, S_2, \ldots$. Simply let $\bar{e}$ be the lowest e for which R_e is identical to R, and apply Lemma 3. But no set occurs more than once in the sequence $S_0, S_1, \ldots$, by Lemmas 4 and 5. The theorem is proved.

COROLLARY TO THEOREM 3. *There exists a sequence of uniformly partial recursive functions which contains every partial recursive function once and only once.*

PROOF. Instead of letting the R's be all recursively enumerable sets, let them be all partial recursive functions, expressed as sets of ordered

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pairs. Construct the S's as in Theorem 3, except that in Act C of Case 3 the new member of $S_{x'}$ must be an ordered pair which differs *in its argument member* from any ordered pair previously listed in any R or S. The S's will be the desired sequence of partial recursive functions expressed as sets of ordered pairs.

BIBLIOGRAPHY

[1] J. R. MYHILL, this JOURNAL, vol. 21, p. 215 (1956), Problems 8 and 9.

[2] EMIL L. POST, *Recursively enumerable sets of positive integers and their decision problems*. ***Bulletin of the American Mathematical Society,*** vol. 50, pp. 284–316 (1944).

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