12. Eilenberg-Zilber Theorem and Künneth Formula

Definition 12.1. Let (C_*, d) and (G_*, ∂) be nonnegative chain complexes. Their **tensor product** $C_* \otimes G_*$ is the (nonnegative) chain complex whose term of degree $n \geq 0$ is

$$(C_* \otimes G_*)_n = \sum_{i+j=n} C_i \otimes G_j .$$

The differentiation $D_n : (C_* \otimes G_*)_n \to (C_* \otimes G_*)_{n-1}$ is defined on generators by

$$D_n(c_i \otimes g_j) = dc_i \otimes g_j + (-1)^i c_i \otimes \partial g_j \quad , \quad \text{where } i+j=n.$$

The $(-1)^i$ in the definition of D_n is needed, since $D_{n-1}D_n$ must be 0:

$$\begin{aligned} D_{n-1}[D_n(c_i \otimes g_j)] &= D_{n-1}(dc_i \otimes g_j + (-1)^i c_i \otimes \partial g_j) \\ &= \underbrace{ddc_i}_{0} \otimes g_j + (-1)^{i-1} dc_i \otimes \partial g_j \\ &\quad + (-1)^i dc_i \otimes \partial g_j + (-1)^i (-1)^i c_i \otimes \underbrace{\partial\partial g_j}_{0} \\ &= 0 . \end{aligned}$$

Theorem 12.2. (Eilenberg-Zilber) Let X and Y be topological spaces. There is a natural chain equivalence

$$\xi : S_*(X \times Y) \to S_*(X) \otimes S_*(Y),$$

unique up to chain homotopy. Thus, for all $n \geq 0$,

$$H_n(X \times Y) \cong H_n(S_*(X) \otimes S_*(Y)).$$

Proof. Let Top×Top be the category whose objects are all ordered pairs of topological spaces (A, B) (B does not need to be a subspace of A) and whose morphisms are all ordered pairs of continuous maps. Composition of morphisms is done coordinatewise.

Let $\mathcal{M}$ be the set of all (Δ^p, Δ^q), $p, q \geq 0$.

Define

$$F : \text{Top}\times\text{Top} \to \text{Comp}, \quad (X,Y) \mapsto S_*(X\times Y),$$

$$E : \text{Top}\times\text{Top} \to \text{Comp}, \quad (X,Y) \mapsto S_*(X) \otimes S_*(Y).$$

Recall: A category $\mathcal{C}$ with models $\mathcal{M}$ is an ordered pair $(\mathcal{C}, \mathcal{M})$ where $\mathcal{M}$ is a subset of obj $\mathcal{C}$. If $F : \mathcal{C} \to \text{Ab}$ is a functor, then an F-model set is an indexed set $X = \{x_j \in FM_j \mid j \in J\}$, where $\{M_j \mid j \in J\}$ is an indexed family of models. The functor F is free with base in $\mathcal{M}$, if

1) FC is a free abelian group $\forall\, C \in$ obj $\mathcal{C}$,

2) There is an F-model set $X = \{x_j \in FM_j \mid j \in J\}$ with the property that $\forall\, C \in$ obj $\mathcal{C}$ the set

$$\{(F\sigma)(x_j) \mid x_j \in X,\ \sigma : M_j \to C\}$$

is a basis for FC.

Also, $C \in$ obj $\mathcal{C}$ is called F-acyclic, if $H_n(FC) = 0$, for all $n > 0$, where $F : \mathcal{C} \to \text{Comp}$.

Claim 1. F is free and (Δ^p, Δ^p) is F-acyclic $\forall\, p \geq 0$.
(also find a base for F_p $\forall\, p\geq 0$)

Proof of Claim 1. Let $p \geq 0$. Let

$$d^p : \Delta^p \to \Delta^p \times \Delta^p, \quad x \mapsto (x,x).$$

Then $d^p \in S_p(\Delta^p \times \Delta^p) = F_p(\Delta_p, \Delta_p)$.

Let $F_p : Top \times Top \to Ab$, $(X,Y) \mapsto S_p(X \times Y)$.
Let $\mathcal{M}_p = \{ \underbrace{\Delta^p \times \Delta^p}_{M_p} \} \subset obj(Top \times Top)$.
Define an F_p-model set $X_p = \{d_p\}$.
Let $A_1 \times A_2 \in Top \times Top$.

First, let $\sigma : M_p \to A_1 \times A_2$ be a morphism. Then σ is a continuous map

$$\sigma = (\sigma_1, \sigma_2) : \Delta_p \times \Delta_p \to A_1 \times A_2,$$

and σ induces

$$\begin{array}{ccccc} F_p\sigma : & F_p(\Delta_p \times \Delta_p) & \longrightarrow & F_p(A_1 \times A_2) & , \ \phi \mapsto \sigma \circ \phi . \\ & \| & & \| & \\ & S_p(\Delta_p \times \Delta_p) & & S_p(A_1 \times A_2) & \end{array}$$

Then

$$(F_p\sigma)(d_p) \in S_p(A_1 \times A_2),$$

$$\begin{array}{rccccc} (F_p\sigma)(d_p) : & \Delta_p & \xrightarrow{d_p} & \Delta^p \times \Delta^p & \xrightarrow{\sigma} & A_1 \times A_2 \\ & x & \mapsto & (x,x) & \mapsto & (\sigma_1(x), \sigma_2(x)) \end{array}$$

Thus

$$\{ (F_p\sigma)(d_p) \mid \sigma : M_p \to A_1 \times A_2 \} = \{ \phi : \Delta_p \to A_1 \times A_2 \mid \phi \text{ cont} \}$$

$\Rightarrow \{ (F_p\sigma)(d_p) \mid \sigma : M_p \to A_1 \times A_2 \}$ is a basis for $S_p(A_1 \times A_2)$.

Since $S_p(A_1 \times A_2)$ is a free abelian group, it follows that F_p is free with base X_p.

Since $\Delta_p \times \Delta_p$ is convex, it is contractible. Thus $H_n(\Delta_p \times \Delta_p) = 0 \ \forall \ n > 0$. Thus (Δ_p, Δ_p) is F-acyclic.

$$\begin{array}{c} H_n(\Delta_p \times \Delta_p) \\ \| \\ H_n(S_*(\Delta_p \times \Delta_p)) \\ \| \\ H_n(F(\Delta_p \times \Delta_p)) \end{array}$$

$\therefore$ Claim 1

Claim 2. E is free and (Δ_p, Δ_q) is E-acyclic $\forall p,q \geq 0$.
(also find a basis for E_p $\forall p$)

proof of Claim 2.

$S_p(X)$: free abelian, basis all cont. $\sigma : \Delta^p \to X$

$S_q(Y)$: free abelian, basis all cont. $\tau : \Delta^q \to Y$

Then (check this):

$S_p(X) \otimes S_q(Y)$: free abelian with basis $\sigma \otimes \tau$
(σ, τ as above).

Earlier example: S_p is free with base $\{\delta^p\}$,
where $\delta^p = id : \Delta^p \to \Delta^p$.

Let

$$E_n : \mathrm{Top} \times \mathrm{Top} \to \mathcal{A}b,$$

$$(X,Y) \mapsto \left(S_*(X) \otimes S_*(Y)\right)_n = \sum_{p+q=n} S_p(X) \otimes S_q(Y).$$

Then E_n is free with basis in $\mathcal{M}$:

Clearly, E_n is free. Let $\mathcal{Y}_n = \{ \delta^p \otimes \delta^q \in S_p(\Delta^p) \otimes S_q(\Delta^q) \mid p+q=n \}$.

Let $f \times g : \Delta^p \times \Delta^q \to X \times Y$ be continuous.
Then

$$E_n(f\times g) = \sum_{i+j=n} f_i \otimes g_j : \underbrace{\sum_{i+j=n} S_i(\Delta^p) \otimes S_j(\Delta^q)}_{\substack{\parallel \\ E_n(\Delta^p \times \Delta^q) \\ \uparrow \\ \text{means } (\Delta^p, \Delta^q)}} \to \underbrace{\sum_{i+j=n} S_i(X) \otimes S_j(Y)}_{\substack{\parallel \\ E_n(X\times Y) \\ \uparrow \\ \text{means } (X,Y)}}$$

where $f_i : S_i(\Delta^p) \to S_i(X)$, $\sigma \mapsto f \circ \sigma$
$g_j : S_j(\Delta^q) \to S_j(Y)$, $\tau \mapsto g \circ \tau$.

Then $\left(E_n(f\times g)\right)(\delta^p \otimes \delta^q) = (f \circ \delta^p) \otimes (g \circ \delta^q) = f \otimes g$.

It follows that $\{ E_n(f\times g)(\delta^p \otimes \delta^q) \mid \delta^p \otimes \delta^q \in \mathcal{Y}_n,\ f\times g : \Delta^p \times \Delta^q \xrightarrow[\text{cont.}]{} X \times Y \}$
is a basis for $E_n(X,Y)$.

(Δ^p, Δ^q) is E-acyclic:

$\forall p \geq 0$: Δ^p is S_*-acyclic. This means that $H_m(\Delta^p) = H_m(S_*(\Delta^p)) = 0 \; \forall m \geq 1$.

Let $\mathbb{Z}_*$ be the chain complex with $\begin{cases} (\mathbb{Z}_*)_0 = \mathbb{Z} \\ (\mathbb{Z}_*)_m = 0, \; m \geq 1. \end{cases}$

Since $S_*(\Delta^p)$ is a free chain complex, it follows that $S_*(\Delta^p)$ is chain equivalent to $\mathbb{Z}_*$ (Thm 8.13). Then

$$E(\Delta^p, \Delta^q) = S_*(\Delta^p) \otimes S_*(\Delta^q)$$

is chain equivalent to $\mathbb{Z}_* \otimes \mathbb{Z}_* \underset{\uparrow \text{check this}}{\cong} \mathbb{Z}_*$.

Then (Δ^p, Δ^q) is E-acyclic.

$\therefore$ Claim 2

It remains to find a natural equivalence $\tau: F \to E$. We will use the acyclic models theorem to show that τ exists.

Let $(A_1, A_2) \in Top \times Top$. Then

$F_0(A_1, A_2) = S_0(A_1 \times A_2)$ is the free abelian group on all ordered pairs (a_1, a_2) where $a_1 \in A_1$, $a_2 \in A_2$

$E_0(A_1, A_2) = (S_*(A_1) \otimes S_*(A_2))_0$ is the free abelian group on all $a_1 \otimes a_2$, $a_1 \in A_1$, $a_2 \in A_2$.

Let $\varphi_{A_1, A_2}: F_0(A_1, A_2) \to E_0(A_1, A_2)$

be the homomorphism for which $(a_1, a_2) \mapsto a_1 \otimes a_2$.

Check: The homomorphisms φ_{A_1,A_2} induce isomorphisms $H_0F(A_1,A_2) \to H_0E(A_1,A_2)$. These isomorphisms form a natural equivalence $\varphi: H_0F \to H_0E$.

The acyclic models theorem $\Rightarrow$ there is a natural chain equivalence $\tau: F \to E$ over φ:

$$\begin{array}{ccccccccc} \cdots & \to & F_1 & \to & F_0 & \to & H_0F & \to & 0 \\ & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \varphi} & & \\ \cdots & \to & E_1 & \to & E_0 & \to & H_0E & \to & 0 \end{array}$$

Also: τ is unique up to chain homotopy. $\square$

<u>Lemma 12.3.</u> Let A_* and G_* be nonnegative chain complexes. Assume every differentiation in A_* is zero. Then

$$H_n(A_* \otimes G_*) \cong \sum_{i \geq 0} H_n(A_i \otimes G^i_*),$$

where $(G^i_*)_n = G_{n-i}$.

<u>proof.</u> Let d: differentiation in A_*,
∂: differentiation in G_*.

Then the differentiation in $A_* \otimes G_*$ is D, where

$$D_n : (A_* \otimes G_*)_n \to (A_* \otimes G_*)_{n-1},$$

$$a_i \otimes g_j \mapsto da_i \otimes g_j + (-1)^i a_i \otimes \partial g_j \quad , \quad i+j=n.$$

$d=0 \Rightarrow$

$$H_n(A_* \otimes G_*) = \frac{\ker D_n}{\operatorname{im} D_{n+1}} = \sum_i \left(\frac{\ker(\mathrm{id} \otimes \partial_{n-i})}{\operatorname{im}(\mathrm{id} \otimes \partial_{n+1-i})} \right)$$

$$\cong \sum_{i \geq 0} H_n(A_i \otimes G^i_*). \quad \square$$

Lemma 12.4. Let C_* and G_* be nonnegative free chain complexes. Consider $H_*(C_*)$ as a chain complex in which every differentiation is zero. Then

$$H_n(C_* \otimes G_*) \cong H_n(H_*(C_*) \otimes G_*).$$

proof. Let

$B_* =$ a subcomplex of C_*, terms boundaries

$Z_* =$ a subcomplex of C_*, terms cycles

Let ∂' be the differentiation in C_*.
Then $\partial'_n | Z_n = 0$ and also $B_n \subset Z_n \Rightarrow \partial'_n | B_n = 0$.
$\Rightarrow$ the differentiations in B_* and Z_* are zero.

Let B^+_* = the chain complex with $(B^+_*)_n = B_{n-1}$, and with zero differentiations.

There are short exact sequence of complexes:

$$0 \to Z_* \xrightarrow{i} C_* \xrightarrow{d} B^+_* \to 0 \qquad (1)$$

$$0 \to B_* \xrightarrow{j} Z_* \xrightarrow{p} H_* \to 0 \qquad (2)$$

Here: i, j inclusions
p = quotient map
$d_n : C_n \to (B^+_*)_n = B_{n-1}$, $x \mapsto \partial'_n x$.

Since each term in G_* is free abelian, there are exact sequences

$$0 \to Z_* \otimes G_* \xrightarrow{i \otimes id} C_* \otimes G_* \xrightarrow{d \otimes id} B^+_* \otimes G_* \to 0 \qquad (3)$$

and

$$0 \to B_* \otimes G_* \xrightarrow{j \otimes id} Z_* \otimes G_* \xrightarrow{p \otimes id} H_* \otimes G_* \to 0 \qquad (4)$$

Since B_{n-1} is free abelian, it follows from Corollary 8.3, that the exact sequences

$$0 \to Z_n \xrightarrow{i_n} C_n \xrightarrow{\partial_n} B_{n-1} \to 0$$

split. Thus there are homomorphisms

$$q_n : C_n \to Z_n \ , \ q_n i_n = id_{Z_n} .$$

Let

$$\varphi_n = p_n q_n : C_n \to Z_n \to H_n .$$

The map $\varphi : C_* \to H_*$ is a chain map:

Let $c \in C_{n+1}$. Then

$$\begin{tikzcd}
C_{n+1} \arrow[r, "q_{n+1}"] \arrow[d, "\partial'"'] & Z_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "0"] & H_{n+1} \arrow[d, "\partial'' = 0"] \\
C_n \arrow[r, "q_n"] & Z_n \arrow[r, "p_n"] & H_n
\end{tikzcd}$$

Let's check that the diagram commutes:

1) $(\partial'' \underbrace{p_{n+1} q_{n+1}}_{\varphi_{n+1}})(c) = 0$ since $\partial'' = 0$.

2) $\varphi_n(\partial' c) = p_n q_n \partial' c$
$= p_n \partial' c$, since $\partial' c \in Z_n$ and $q_n i_n = id_{Z_n}$
$= 0$, since $\partial' c \in B_n$.

$\therefore \varphi$ is a chain map

Consider next the following diagram:

$$\begin{tikzcd}
H_{n+1}(B^+_* \otimes G_*) \arrow[r, "\Delta"] \arrow[d, "\alpha"] & H_n(Z_* \otimes G_*) \arrow[r, "(i\otimes id)_*"] \arrow[d, "id"] & H_n(C_* \otimes G_*) \arrow[r, "(d\otimes id)_*"] \arrow[d, "\beta"] & H_n(B^+_* \otimes G_*) \arrow[r, "\Delta"] \arrow[d, "\alpha'"] & H_{n-1}(Z_* \otimes G_*) \arrow[d, "id"] \\
H_n(B_* \otimes G_*) \arrow[r, "(j\otimes id)_*"'] & H_n(Z_* \otimes G_*) \arrow[r, "(p\otimes id)_*"'] & H_n(H_* \otimes G_*) \arrow[r, "D"'] & H_{n-1}(B_* \otimes G_*) \arrow[r, "(j\otimes id)_*"'] & H_{n-1}(Z_* \otimes G_*)
\end{tikzcd}$$

Top row: induced by (3) $\Rightarrow$ exact, Δ = connecting homom.
Bottom row: induced by (4) $\Rightarrow$ exact, D = connecting homom.

$H_{n+1}(B^+_* \otimes G_*) = H_n(B_* \otimes G_*)$: define α, α' to be identity maps

$\varphi : C_* \to H_*$: Define $\beta = (\varphi \otimes id)_*$.

__Claim:__ every square in the diagram above commutes up to sign.

Then: modify the five lemma $\Rightarrow$ β is an isomorphism, which is the claim of this lemma.

__Check the first square:__

Let ∂ denote the differentiation in G_*.
Let $b_i^+ \otimes g_{n-i}$ be a cycle in $(B^+_*)_i \otimes G_{n-i} = B_{i-1} \otimes G_{n-i}$.
Then

$$0 = d b^+_i \otimes g_{n-i} + (-1)^i b^+_i \otimes \partial g_{n-i}$$

$$= (-1)^i b^+_i \otimes \partial g_{n-i}, \quad \text{since } d b^+_i = 0$$

B_{i-1}, G_{n-i-1} free abelian $\Rightarrow$ $b^+_i = 0$ or $\partial g_{n-i} = 0$.

(But, if $b^+_i \otimes g_{n-i}$ is a generator, then $b^+_i \neq 0$.)

Let $c_i \in C_i$: $d c_i = b^+_i$.

Then

$$\Delta[b_i^+ \otimes g_{n-i}] = \Delta[\partial' c_i \otimes g_{n-i}]$$ ([ ] denotes the homology class)

Let D' denote the differential in $C_* \otimes G_*$. Then

$$D'(c_i \otimes g_{n-i}) = \underbrace{\partial' c_i}_{b_i^+} \otimes g_{n-i} + (-1)^i c_i \otimes \underbrace{\partial g_{n-i}}_{=0} = b_i^+ \otimes g_{n-i}$$

$$\cap$$

$$B_{i-1} \otimes G_{n-i} \subset Z_{i-1} \otimes G_{n-i}$$

Thus $\Delta[\partial' c_i \otimes g_{n-i}] = [b_i^+ \otimes g_{n-i}]$ in $H_n(Z_* \otimes G_*)$.

$$= (j \otimes id)_* \underbrace{[b_i^+ \otimes g_{n-i}]}_{\text{in } H_n(B_* \otimes G_*)} .$$

$\therefore$ The 1st square commutes.

Check: The commutativity of the other squares.

$\square$

Theorem 12.5. (Künneth Theorem)

1) Let C_* and G_* be nonnegative free chain complexes. There are exact sequences, for all n:

$$0 \to \sum_{i+j=n} H_i(C_*) \otimes H_j(G_*) \xrightarrow{\alpha} H_n(C_* \otimes G_*) \to \sum_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(G_*))$$

$$\to 0, \qquad (*)$$

where $\alpha([z_i] \otimes [z'_j]) = [z_i \otimes z'_j]$.

2) The sequence $(*)$ splits:

$$H_n(C_* \otimes G_*) \cong \sum_{i+j=n} H_i(C_*) \otimes H_j(G_*) \oplus \sum_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(G_*)).$$

proof.

Lemma 12.4 $\Rightarrow$

$$H_n(C_* \otimes G_*) \cong H_n(H_*(C_*) \otimes G_*).$$

Lemma 12.3 $\Rightarrow$

$$H_n(H_*(C_*) \otimes G_*) \cong \sum_i H_n(H_i(C_*) \otimes G^i_*).$$

The proof of the Universal Coefficients Theorem for Homology (Theorem 11.6) $\Rightarrow$ for all n and i, there are split exact sequences

$$0 \to H_i(C_*) \otimes H_n(G^i_*) \xrightarrow{\alpha} H_n(H_i(C_*) \otimes G^i_*)$$
$$\to \mathrm{Tor}(H_i(C_*), H_{n-1}(G^i_*)) \to 0.$$

Therefore, there are split exact sequences

$$0 \to H_i(C_*) \otimes H_{n-i}(G_*) \xrightarrow{\alpha} H_n(H_i(C_*) \otimes G^i_*)$$
$$\to \mathrm{Tor}(H_i(C_*), H_{n-i-1}(G_*)) \to 0. \qquad (**)$$

If $i > n$, then $H_n(G^i_*) = H_{n-i}(G_*) = 0$. Then

$$\sum_i H_i(C_*) \otimes H_{n-i}(G_*) = \sum_{i+j=n} H_i(C_*) \otimes H_j(G_*)$$

and

$$\sum_i \mathrm{Tor}(H_i(C_*), H_{n-i-1}(G_*)) = \sum_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(G_*))$$

Take the direct sum of $(**)$ $\Rightarrow$ there is a split exact sequence

$$0 \to \sum_{i+j=n} H_i(C_*) \otimes H_j(G_*) \xrightarrow{\alpha} H_n(C_* \otimes G_*) \to \sum_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(G_*))$$
$$\to 0. \quad \square$$

Theorem 12.6. (Künneth Formula)

Let X and Y be topological spaces, and let $n \geq 0$. There is a split exact sequence

$$0 \to \sum_{i+j=n} H_i(X) \otimes H_j(Y) \xrightarrow{\alpha''} H_n(X \times Y) \to \sum_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0,$$

where $\alpha'' : [z_i] \otimes [z_j'] \mapsto [\xi'(z_i \otimes z_j')]$,

and $\xi' : S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$

is the inverse of the Eilenberg-Zilber chain equivalence (see Theorem 12.2). Thus

$$H_n(X \times Y) \cong \sum_{i+j=n} H_i(X) \otimes H_j(Y) \oplus \sum_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)).$$

proof. Künneth Theorem $\Rightarrow$ there is a split exact sequence

$$0 \to \sum_{i+j=n} H_i(X) \otimes H_j(Y) \xrightarrow{\alpha} H_n(S_*(X) \otimes S_*(Y))$$

$$\to \sum_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0.$$

Eilenberg-Zilber Theorem $\Rightarrow$

$$H_n(S_*(X) \otimes S_*(Y)) \cong H_n(S_*(X \times Y)) = H_n(X \times Y).$$

$\square$

Example.

Let $m, n \in \mathbb{Z}$, $m, n > 0$.

1) Assume $m \neq n$. Then

$$H_p(S^m \times S^n) = \sum_{i+j=p} H_i(S^m) \otimes H_j(S^n) \oplus \overbrace{\sum_{r+s=p-1} \mathrm{Tor}(H_r(S^m), \underset{\uparrow\, 0,\ \text{or}\ \mathbb{Z}}{H_s(S^n)})}^{=0 \text{ by Tor 2}}$$

$$= \begin{cases} \mathbb{Z}, & \text{if } (i,j) \in \{(0,n), (m,0), (m,n), (0,0)\},\ i+j=p \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \mathbb{Z}, & \text{if } p = 0, m, n, m+n \\ 0, & \text{otherwise} \end{cases}$$

2) Assume $m = n$.

Again, $\mathrm{Tor}(H_r(S^m), H_s(S^m)) = 0 \ \forall r, s$.

$$H_p(S^m \times S^m) = \begin{cases} \mathbb{Z}, & \text{if } p = 0, 2m \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } p = m \\ 0, & \text{otherwise} \end{cases}$$

3) Let $X = S^1 \vee S^2 \vee S^3$. Then

$$H_p(X) = \begin{cases} \mathbb{Z}, & \text{if } p = 0, 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

Thus, by part 1, $H_p(X) = H_p(S^1 \times S^2)$, for all p.

One can show that X and $S^1 \times S^2$ do not have the same homotopy type.

<u>Example.</u> Let $X = \mathbb{R}P^3 \times \mathbb{R}P^2$.

Theorem 6.2 $\Rightarrow$ $\underset{(3 \text{ odd})}{H_p(\mathbb{R}P^3)} = \begin{cases} \mathbb{Z}, & \text{if } p = 0 \text{ or } p = 3 \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } p = 1 \\ 0, & \text{otherwise} \end{cases}$

and

$$\underset{(2 \text{ even})}{H_p(\mathbb{R}P^2)} = \begin{cases} \mathbb{Z}, & \text{if } p = 0 \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } p = 1 \\ 0, & \text{otherwise} \end{cases}$$

Künneth Formula $\Rightarrow$

$$H_p(\mathbb{R}P^3 \times \mathbb{R}P^2) = \sum_{i+j=p} H_i(\mathbb{R}P^3) \otimes H_j(\mathbb{R}P^2) + \sum_{r+s=p-1} \mathrm{Tor}(H_r(\mathbb{R}P^3), H_s(\mathbb{R}P^2))$$

$\underline{p=0:}$ $i=j=0$

$H_0(\mathbb{R}P^3) \otimes H_0(\mathbb{R}P^2) = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$

no Tor-term $\Rightarrow$ $H_0(\mathbb{R}P^3 \times \mathbb{R}P^2) = \mathbb{Z}$

$\underline{p=1:}$ $i=1,\ j=0$

$H_1(\mathbb{R}P^3) \otimes H_0(\mathbb{R}P^2) = (\mathbb{Z}/2\mathbb{Z}) \otimes \mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$

Tor: $r+s = p-1 = 1-1 = 0 \Rightarrow r = s = 0 \Rightarrow \mathrm{Tor}(H_0(\mathbb{R}P^3), H_0(\mathbb{R}P^2))$
$= \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}) = 0$

$i=0,\ j=1$

$H_0(\mathbb{R}P^3) \otimes H_1(\mathbb{R}P^2) = \mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, again $\mathrm{Tor} = 0$

$\Rightarrow$ $H_1(\mathbb{R}P^3 \times \mathbb{R}P^2) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$

$\underline{p=2:}$ $i=0,\ j=2$

$H_0(\mathbb{R}P^3) \otimes H_2(\mathbb{R}P^2) = \mathbb{Z} \otimes 0 = 0$

$i=1,\ j=1$: $H_1(\mathbb{R}P^3) \otimes H_1(\mathbb{R}P^2) = (\mathbb{Z}/2\mathbb{Z}) \otimes (\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

$i=2,\ j=0$: $\underbrace{H_2(\mathbb{R}P^3)}_{0} \otimes H_0(\mathbb{R}P^2) = 0$

Tor-terms: $r+s = 2-1 = 1$

$r=0,\ s=1$: $\mathrm{Tor}(H_0(\mathbb{R}P^3), H_1(\mathbb{R}P^2)) = \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \overset{\mathrm{Tor}\,2}{=} 0$

$r=1,\ s=0$: $\mathrm{Tor}(H_1(\mathbb{R}P^3), \underbrace{H_0(\mathbb{R}P^2)}_{\mathbb{Z}}) = 0$ (Tor 2)

$\Rightarrow$ $H_2(\mathbb{R}P^3 \times \mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$.

$\underline{p=3:}$ $i=3,\ j=0$

$$H_3(\mathbb{R}P^3) \otimes H_0(\mathbb{R}P^2) = \mathbb{Z}\otimes\mathbb{Z} = \mathbb{Z}$$

$i=2,\ j=1$: $\underbrace{H_2(\mathbb{R}P^3)}_{0} \otimes H_1(\mathbb{R}P^2) = 0$

$i=1,\ j=2$: $H_1(\mathbb{R}P^3) \otimes \underbrace{H_2(\mathbb{R}P^2)}_{0} = 0$

$i=0,\ j=3$: $H_0(\mathbb{R}P^3) \otimes \underbrace{H_3(\mathbb{R}P^2)}_{0} = 0$

Tor-terms : $r+s = 3-1 = 2$

$r=2, s=0$: $\mathrm{Tor}(H_2(\mathbb{R}P^3), H_0(\mathbb{R}P^2)) = \mathrm{Tor}(0, \mathbb{Z}) = 0$

$r=1, s=1$: $\mathrm{Tor}(H_1(\mathbb{R}P^3), H_1(\mathbb{R}P^2)) = \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$

$$\overset{T4}{=} \{ k \in \mathbb{Z}/2\mathbb{Z} \mid 2k = 0 \} = \mathbb{Z}/2\mathbb{Z}$$

$r=0, s=2$: $\mathrm{Tor}(H_0(\mathbb{R}P^3), \underbrace{H_2(\mathbb{R}P^2)}_{0}) = 0$

$\Rightarrow H_3(\mathbb{R}P^3 \times \mathbb{R}P^2) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$

$\underline{p=4:}$
$i=4,\ j=0$: $H_4(\mathbb{R}P^3) = 0 \to 0$
$i=3,\ j=1$: $H_3(\mathbb{R}P^3) \otimes H_1(\mathbb{R}P^2) = \mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$
$i=2,\ j=2$: $H_2(\mathbb{R}P^2) = 0 \to 0$
$i=1,\ j=3$: $H_3(\mathbb{R}P^2) = 0 \to 0$
$i=0,\ j=4$: $\to 0$

Tor-terms : $r+s = 4-1 = 3$

$r=3, s=0$: $\mathrm{Tor}(H_3(\mathbb{R}P^3), \overbrace{H_0(\mathbb{R}P^2)}^{\mathbb{Z}}) \overset{T2}{=} 0$
$r=2, s=1$: $\mathrm{Tor}(0, \ ) = 0$
$r=1, s=2$: $\mathrm{Tor}(\ , 0) = 0$
$r=0, s=3$: $\mathrm{Tor}(\mathbb{Z}, \ ) = 0$ Thus $H_4(\mathbb{R}P^3 \times \mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$

$\underline{p>4}$: $H_i(\mathbb{R}P^3) \otimes H_j(\mathbb{R}P^2)$: $i+j = p > 4$

if $j > 1$, then $H_j(\mathbb{R}P^2) = 0 \Rightarrow H_i(\mathbb{R}P^3) \otimes H_j(\mathbb{R}P^2) = 0$.

if $j \leq 1$, then $i = p - j > 4 - 1 = 3 \Rightarrow H_i(\mathbb{R}P^3) = 0$

$\Rightarrow H_i(\mathbb{R}P^3) \otimes H_j(\mathbb{R}P^2) = 0$.

Tor-terms: $\mathrm{Tor}(H_r(\mathbb{R}P^3), H_s(\mathbb{R}P^2))$, $r+s = p-1 > 4-1 = 3$

Tor 2 $\Rightarrow \mathrm{Tor}(H_r(\mathbb{R}P^3), H_s(\mathbb{R}P^2)) = 0$ if $(r,s) \neq (1,1)$.

Then $r+s = 3 \Rightarrow \mathrm{Tor}(H_r(\mathbb{R}P^3), H_s(\mathbb{R}P^2)) = 0$

$\Rightarrow H_p(\mathbb{R}P^3 \times \mathbb{R}P^2) = 0$ for all $p > 4$.

13. Cohomology groups

For any abelian groups G'', G', let $\mathrm{Hom}(G'', G')$ denote the set of all group homomorphisms $G'' \to G'$. Let G be an abelian group. Then

$$\mathrm{Hom}(\ , G) : \mathrm{Ab} \to \mathrm{Ab}$$

is a contravariant functor:

$(\mathrm{Hom}(\ , G))(H) = \mathrm{Hom}(H, G)$ is an abelian group

(group operation: Let $f, g \in \mathrm{Hom}(H, G)$, then $f+g \in \mathrm{Hom}(H, G)$, where $(f+g)(x) = f(x) + g(x)$)

Let A, B be abelian groups, and let $\varphi: A \to B$ be a homomorphism. Then φ induces a homomorphism

$\varphi^\# : \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G), \ f \mapsto f\varphi.$

$\big(\varphi^\#(f+g) = (f+g)\varphi = f\varphi + g\varphi,$ let $f : B \to G,\ f(y) = 0 \ \forall y \in B,$
then $(\varphi^\# f)(x) = (f\varphi)(x) = f(\varphi(x)) = 0 \ \forall x \in A \big)$

Let $\varphi = \mathrm{id} : A \to A$. Then $\varphi^\# : \mathrm{Hom}(A,G) \to \mathrm{Hom}(A,G),$
$f \mapsto \varphi f = f$

$$\Rightarrow \varphi^\# = \mathrm{id}_{\mathrm{Hom}(A,G)}.$$

Let $\varphi : A \to B,\ \psi : B \to C$ be homomorphisms between abelian groups. Then

$$\varphi^\# : \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G),\ f \mapsto f\varphi$$

and

$$\psi^\# : \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G),\ g \mapsto g\psi$$

Thus $\varphi^\# \circ \psi^\# : \mathrm{Hom}(C,G) \xrightarrow{\psi^\#} \mathrm{Hom}(B,G) \xrightarrow{\varphi^\#} \mathrm{Hom}(A,G)$
$g \mapsto g\psi \mapsto g\psi\varphi = (\psi\circ\varphi)^\# g$

$\therefore \mathrm{Hom}(\ ,G)$ is a contravariant functor

Hom(, G) is an additive functor:

Let $\varphi, \psi : A \to B$ be homomorphisms. Then $\varphi + \psi : A \to B$ is a homomorphism, and

$$(\varphi+\psi)^\# : \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G),$$

$(\varphi+\psi)^\#(f) = f(\varphi+\psi) = f\varphi + f\psi$, since f is a homomorphism

$$= \varphi^\# f + \psi^\# f$$

$\therefore (\varphi+\psi)^\# = \varphi^\# + \psi^\#$

Lemma 13.1. Let X be a topological space, and let G be any abelian group. Let $(S_*(X), \partial)$ be the singular chain complex of X. Then

$$0 \to \mathrm{Hom}(S_0(X), G) \xrightarrow{\partial_1^\#} \mathrm{Hom}(S_1(X), G) \xrightarrow{\partial_2^\#} \mathrm{Hom}(S_2(X), G) \to \cdots$$

is a complex, denoted by $\mathrm{Hom}(S_*(X), G)$.

proof. Let $n \geq 1$. Then

$$\partial_{n+1}^\# \partial_n^\# = (\partial_n \partial_{n+1})^\# = 0^\# = 0. \quad \square$$

Let G be an abelian group. Let F be a free abelian group with basis B. Let $\varphi \in \mathrm{Hom}(F, G)$. Then $\varphi| : B \to G$ is a function. Let $f : B \to G$ be a function. Then f defines a homomorphism (unique) $\varphi_f : F \to G$.

Write $\mathrm{Hom}(S_n(X), G) = A_{-n}$ and $\partial_{n+1}^\# = d_{-n}$. Then the sequence in Lemma 13.1 becomes

$$0 \to A_0 \xrightarrow{d_0} A_{-1} \xrightarrow{d_{-1}} A_{-2} \xrightarrow{d_{-2}} \cdots \to A_{-n+1} \xrightarrow{d_{-n+1}} A_{-n} \xrightarrow{d_{-n}} A_{-n-1} \to \cdots$$

Thus (A_*, d) is a complex whose all nonzero terms have nonpositive degree. Define the cycles, boundaries and homology for $\mathrm{Hom}(S_*(X), G)$:

- $-n$ cycles: $\ker d_{-n}$
- $-n$ boundaries: $\mathrm{im}\, d_{-n+1}$

$$H_{-n}(\mathrm{Hom}(S_*(X), G)) = H_{-n}(A_*) = \ker d_{-n} / \mathrm{im}\, d_{-n+1} = \ker \partial_{n+1}^\# / \mathrm{im}\, \partial_n^\# .$$