2) Assume both F and E are free and acyclic. Then there are augmentation preserving natural chain maps $\tau: F \to E$ and $\sigma: E \to F$. The identity chain map $id: F \to F$ is also augmentation preserving. Uniqueness $\Rightarrow$ there is a natural chain homotopy $\sigma\tau \simeq id_F$ and a natural chain homotopy $\tau\sigma \simeq id_E$. Then τ and σ are natural chain equivalences. $\square$

10. Tensor products

Definition 10.1. Let A and B be abelian groups. The tensor product $A \otimes B$ of A and B is the abelian group having the following presentation:

Generators: all ordered pairs $(a,b) \in A \times B$.

Relations: $(a+a', b) = (a,b) + (a',b)$ and
$(a, b+b') = (a,b) + (a,b')$, for all $a, a' \in A$ and $b, b' \in B$.

Let F be the free abeliangroup generated by all ordered pairs $(a,b) \in A \times B$. Let N be the subgroup of F generated by all relations, i.e., by elements of the form $(a+a', b) - (a,b) - (a',b)$ or of the form $(a, b+b') - (a,b) - (a,b')$. Then $A \otimes B = F/N$. Denote the coset $(a,b) + N$ by $a \otimes b$. Then a typical element in $A \otimes B$ has an expression of the form $\sum m_i (a_i \otimes b_i)$ for $m_i \in \mathbb{Z}$.

Exercises

1) $a \otimes 0 = 0 = 0 \otimes b$ for all $a \in A$ and $b \in B$.

2) If $m \in \mathbb{Z}$, then $m(a \otimes b) = (ma) \otimes b = a \otimes (mb)$.

3) If A is torsion, then $A \otimes \mathbb{Q} = 0$ (Hint: If $a \in A$, then $ma = 0$, for some $m > 0$; if $q \in \mathbb{Q}$, then $a \otimes q = a \otimes m(q/m) = ma \otimes (q/m) = 0$.)

4) If A and B are finite abelian groups whose orders are relatively prime, then $A \otimes B = 0$.

Definition 10.2. Let A, B and G be abelian groups. A *bilinear function* $\varphi : A \times B \to G$ is a function such that

$$\varphi(a + a', b) = \varphi(a, b) + \varphi(a', b)$$

and

$$\varphi(a, b + b') = \varphi(a, b) + \varphi(a, b'),$$

for all $a, a' \in A$ and all $b, b' \in B$.

The map $\iota : A \times B \to A \otimes B$, $(a, b) \mapsto a \otimes b$, is bilinear.

Theorem 10.3.

1) Let G be any abelian group, and let $\varphi : A \times B \to G$ be any bilinear map. Then there is a unique homomorphism $f : A \otimes B \to G$ making the following diagram commute

$$\begin{tikzcd} A \times B \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & A \otimes B \arrow[d, dashed, "f"] \\ & G \end{tikzcd}$$

(Here: $\iota : A \times B \to A \otimes B$
$(a, b) \mapsto a \otimes b$)

2) $A \otimes B$ is the only group with this property, this means that if T is an abelian group and if $\eta : A \times B \to T$ is a bilinear map s.t. the diagram

$$\begin{tikzcd} A \times B \arrow[r, "\eta"] \arrow[dr, "\varphi"'] & T \arrow[d, "f"] \\ & G \end{tikzcd}$$

always has a unique "completion" f, then $T \cong A \otimes B$

<u>proof.</u> 1) Write $A \otimes B = F/N$, where F is a free abelian group with basis $A \times B$ and N is generated by the relations as in Definition 10.1. Consider the diagram

$$\begin{tikzcd} A \times B \arrow[r, hook] \arrow[dr, "\varphi"'] & F \arrow[r] \arrow[d, "\tilde{\varphi}"] & F/N = A \otimes B \arrow[dl, dashed, "f"] \\ & G & \end{tikzcd}$$

The homomorphism $\tilde{\varphi}$ is induced by φ (extend by linearity). Now φ bilinear $\Rightarrow N \subset \ker \tilde{\varphi}$. Thus $\tilde{\varphi}$ induces a homomorphism

$$f : A \otimes B \to G, \quad (a,b) + N \mapsto \tilde{\varphi}(a,b).$$

If $(a,b) \in A \times B$, then $f(a \otimes b) = \tilde{\varphi}(a,b) = \varphi(a,b)$. Since the elements $a \otimes b$, $(a,b) \in A \times B$, generate $A \otimes B$, it follows that f is unique. $\square$

2) Consider the diagram:

$$\begin{tikzcd}
A \times B \arrow[r, "\iota"] \arrow[dr, "\psi"'] & A \otimes B \arrow[d, shift left, "\varphi"'] \\
& T \arrow[u, shift left, "g"']
\end{tikzcd}$$

By assumption, there are homomorphisms

$\varphi : A \otimes B \to T$ and $g : T \to A \otimes B$

with $\varphi\iota = \psi$ and $g\psi = \iota$.

Consider the diagram

$$\begin{tikzcd}
A \times B \arrow[r, "\iota"] \arrow[d, "\iota"'] & A \otimes B \arrow[dl, dashed] \\
A \otimes B &
\end{tikzcd}$$

Letting the map $A \otimes B \to A \otimes B$ be either the identity or $g\varphi$ will make the diagram commute ($g\varphi\iota = g(\varphi\iota) = g\psi = \iota$). Since the completion is unique it follows that $g\varphi = id_{A\otimes B}$. Using a similar diagram shows that $\varphi g = id_T$. Thus φ and g are isomorphisms.

$\square$

Theorem 10.4. Let A, A', B, B' be abelian groups, and let $f: A \to A'$ and $g: B \to B'$ be homomorphisms.

1) There is a unique homomorphism $f \otimes g: A \otimes B \to A' \otimes B'$ satisfying
$$(f \otimes g)(a \otimes b) = fa \otimes gb \qquad \forall a \in A,\ b \in B.$$

2) Let $f': A' \to A''$ and $g': B' \to B''$ be homomorphisms, where A'', B'' are abelian groups. Then
$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g).$$

proof. 1) Let
$$\varphi: A \times B \to A' \otimes B', \quad (a,b) \mapsto fa \otimes gb.$$

Then φ is bilinear. Theorem 10.3.1 $\Rightarrow$ $\exists$ unique homomorphism $f \otimes g: A \otimes B \to A' \otimes B'$ making the following diagram commute:

$$\begin{tikzcd} A \times B \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & A \otimes B \arrow[d, dashed, "f \otimes g"] \\ & A' \otimes B' \end{tikzcd}$$

$\square$

2) Consider the diagram

$$\begin{tikzcd} A \times B \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & A \otimes B \arrow[d, dashed] \\ & A'' \otimes B'' \end{tikzcd}$$

where $\varphi(a,b) = f'(f(a)) \otimes g'(g(b))$. Then

$$((f' \circ f) \otimes (g' \circ g)) \circ \iota = \varphi = (f' \otimes g') \circ (f \otimes g) \circ \iota.$$

Uniqueness (Thm 10.3.1) $\Rightarrow$ $(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g)$.

$\square$

Corollary 10.5. Let A be an abelian group. There is a functor

$$T = T_A : \mathcal{Ab} \to \mathcal{Ab}, \quad \text{with}$$

$$T(B) = A \otimes B \quad \text{and} \quad T(\phi) = id_A \otimes \phi.$$

proof. Let B, B', B'' be abelian groups. Let

$$\phi : B \to B' \quad \text{and} \quad \phi' : B' \to B''$$

be homomorphisms. Theorem 10.4.2 $\Rightarrow$

$$T(\phi' \circ \phi) = id_A \otimes (\phi' \circ \phi) = (id_A \otimes \phi') \circ (id_A \otimes \phi) = T(\phi') \circ T(\phi).$$

Also, $T(id_B) = id_A \otimes id_B = id_{A \otimes B}$,

by Thm 10.3.1. $\square$

Similarly, we have:

Corollary 10.6: Let B be an abelian group. There is a functor

$$F = F_B : \mathcal{Ab} \to \mathcal{Ab}, \quad \text{with}$$

$$F(A) = A \otimes B \quad \text{and} \quad F(g) = g \otimes id_B.$$

$\square$

The functors T and F are often denoted by, $A \otimes -$ and $- \otimes B$, respectively. $\square$

Exercises:

1) a) Prove that there is an isomorphism

$$A \otimes B \to B \otimes A, \quad a \otimes b \mapsto b \otimes a.$$

b) Prove that, for any abelian group, the functors $A \otimes -$ and $- \otimes A$ are isomorphic.

2) Prove that the tensor product functor T_A (and F_B) is additive. Conclude that if $f: B \to B'$ is the zero map ($f = 0$), then $T(f) = 0$, and if $B = \{0\}$, then $T(B) = \{0\}$.

3) Let $f: B \to B$ be multiplication by an integer m, $f(b) = mb$ for all $b \in B$. Show that $id_A \otimes f : A \otimes B \to A \otimes B$ is also multiplication by m.

4) a) Show that, for every abelian group A, there is an isomorphism $J_A : \mathbb{Z} \otimes A \to A$, $n \otimes a \mapsto na$.

b) Show that the family of all J_A comprise a natural equivalence between $\mathbb{Z} \otimes -$ and the identity functor on Ab.

Theorem 10.7. Let A and B_j, $j \in J$, be abelian groups. There is an isomorphism

$$A \otimes \sum B_j \to \sum (A \otimes B_j), \quad \leftarrow$$

Elements in $\sum B_j$: (b_j) where $b_j \neq 0$ for only finitely many j.

with $a \otimes (b_j) \mapsto (a \otimes b_j)$.

proof. The function

$$\eta : A \times \sum B_j \to \sum (A \otimes B_j), \quad (a, (b_j)) \mapsto (a \otimes b_j),$$

is bilinear. Let G be an abelian group, and

Let $\varphi : A \times \sum B_j \to G$ be a bilinear function. Consider the diagram

$$\begin{tikzcd} A \times \sum B_j \arrow[rr, "\eta"] \arrow[dr, "\varphi"'] & & \sum(A \otimes B_j) \arrow[dl, dashed] \\ & G & \end{tikzcd}$$

Let $\bar{b}_j \in \sum B_j$ be the element that has b_j in the j^{th} coordinate and 0 elsewhere. For every $j \in J$, let

$$\varphi_j : A \times B_j \to G, \ (a, b_j) \mapsto \varphi(a, \bar{b}_j).$$

Clearly, φ_j is bilinear. Thus, by Theorem 10.3, there is a homomorphism $f_j : A \otimes B_j \to G$ making the following diagram commute:

$$\begin{tikzcd} A \times B_j \arrow[rr, "\nu_j"] \arrow[dr, dash, "\varphi_j"'] & & A \otimes B_j \arrow[dl, dash, "f_j"] \\ & G & \end{tikzcd}$$

$$f_j(a \otimes b_j) = \varphi_j(a, b_j) = \varphi(a, \bar{b}_j).$$

<u>Fact:</u> Let $\{B_j \mid j \in J\}$ be a family of abelian groups, let G be an abelian group, and let $\{f_j : B_j \to G \mid j \in J\}$ be a family of homomorphisms. Then there is a unique homomorphism $f : \sum B_j \to G$ with $f|B_j = f_j$, for all $j \in J$. $\square$

By the above fact, there is a homomorphism

$$f : \sum(A \otimes B_j) \to G,$$

$$\sum a \otimes b_j \mapsto \sum f_j(a \otimes b_j) = \sum \varphi(a, \bar{b}_j) = \varphi(a, \sum \bar{b}_j) = \varphi(a, (b_j)).$$

Thus $f \circ \eta = \varphi$. By Theorem 10.3.2, $\sum(A \otimes B_j) \cong A \otimes \sum B_j$.

Consider the diagram

$$\begin{tikzcd}
A \times \Sigma B_j \arrow[rr, "\nu"] \arrow[dr, "\eta"'] & & A \otimes \Sigma B_j \arrow[dl, "h"] \\
& \Sigma (A \otimes B_j) &
\end{tikzcd}$$

By Theorem 10.3.2, the homomorphism h making the diagram commute is an isomorphism. Clearly, $h(a \otimes (e_j)) = (a \otimes e_j)$. $\square$

11. Universal coefficients

Let (C_*, ∂) be a chain complex, and let G be an abelian group. Then $(C_* \otimes G, \partial \otimes id_G)$ is a chain complex:

$$\cdots \to C_{n+1} \otimes G \xrightarrow{\partial_{n+1} \otimes id_G} C_n \otimes G \xrightarrow{\partial_n \otimes id_G} C_{n-1} \otimes G \to \cdots$$

$$(\partial_n \otimes id_G) \circ (\partial_{n+1} \otimes id_G) = (\partial_n \circ \partial_{n+1}) \otimes id_G = 0 \otimes id_G = 0,$$

since $(0 \otimes id_G)(x, g) = 0 \times g = 0$.

__Definition 11.1.__ Let (X, A) be a pair of topological spaces. Let G be an abelian group. Let $(S_*(X, A), \partial)$ be the singular chain complex of (X, A). The singular chain complex of (X, A) with __coefficients__ G is the complex

$$\cdots \to S_{n+1}(X, A) \otimes G \xrightarrow{\partial \otimes id} S_n(X, A) \otimes G \xrightarrow{\partial \otimes id} S_{n-1}(X, A) \otimes G \to \cdots$$

The n^{th} __homology group__ of (X, A) __with coefficients G__ is

$$H_n(X, A; G) = \ker(\partial_n \otimes id) / \mathrm{im}(\partial_{n+1} \otimes id).$$

Homology with coefficients has to do with <u>spectral sequences</u>. Spectral sequences are important in computing homology groups. The calculations have terms of the form $H_p(X, H_q(Y))$.

Next, we should find out how tensoring with G affects exact sequences.

<u>Theorem 11.2.</u> Let A be an abelian group, and let

$$B' \xrightarrow{i} B \xrightarrow{p} B'' \to 0$$

be an exact sequence of abelian groups. Then there are exact sequences

$$A \otimes B' \xrightarrow{id \otimes i} A \otimes B \xrightarrow{id \otimes p} A \otimes B'' \to 0$$

and

$$B' \otimes A \xrightarrow[i \otimes id]{} B \otimes A \xrightarrow[p \otimes id]{} B'' \otimes A \to 0$$

<u>Proof.</u> We prove the exactness of the first sequence. The proof for the second sequence is similar.

1) $\operatorname{im}(id \otimes i) \subset \ker(id \otimes p)$: It suffices to prove that $(id \otimes p) \circ (id \otimes i) = 0$.

Now, $(id \otimes p) \circ (id \otimes i) = id \otimes (p \circ i)$
$= id \otimes 0 = 0$

2) $\ker(id \otimes p) \subset \operatorname{im}(id \otimes i)$:

Let $E = \operatorname{im}(id \otimes i)$. Since $E \subset \ker(id \otimes p)$, the homomorphism

$$\bar{p} : (A \otimes B)/E \to A \otimes B''$$

$$a \otimes b + E \mapsto a \otimes pb$$

is well-defined. Let

$$\pi : A \otimes B \to (A \otimes B)/E, \quad a \otimes b \mapsto a \otimes b + E.$$

Then
$$\bar{p} \circ \pi = id \otimes p : A \otimes B \to A \otimes B''.$$

Assume $\bar{p}$ is an isomorphism. Then

$$\ker(id \otimes p) = \ker(\bar{p} \circ \pi) = \pi^{-1}(\bar{p}^{-1}(0)) = \pi^{-1}(0) = \ker \pi$$

$$= E = \operatorname{im}(id \otimes i).$$

Thus it suffices to show that $\bar{p}$ is an isomorphism. Let

$$f : A \times B'' \to (A \otimes B)/E, \quad (a, b'') \mapsto a \otimes b + E,$$

where $pb = b''$.

f is well-defined: p is a surjection $\Rightarrow \exists\, b \in B : pb = b''$.

Assume that also $pb_1 = b''$. Then $p(b - b_1) = p(b) - p(b_1) = 0$

$\Rightarrow b - b_1 \in \ker p = \operatorname{im} i$.

$\Rightarrow b - b_1 = ib'$, for some $b' \in B'$.

Thus
$$a \otimes b - a \otimes b_1 = a \otimes (b - b_1) = (id \otimes i)(a \otimes b') \in \operatorname{im}(id \otimes i) = E.$$

$\therefore f$ is well-defined

Clearly, f is bilinear. By Theorem 10.3.1, there is a homomorphism $\bar{f}$ making the following diagram commute:

$$\begin{tikzcd} A \times B'' \arrow[r, "\nu"] \arrow[dr, "d"'] & A \otimes B'' \arrow[d, dashed, "\bar{d}"] \\ & (A\otimes B)/E \end{tikzcd}$$

Here, $\bar{d}(a \otimes b'') = d(a, b'') = a \otimes b + E$, where $p(b) = b''$.

Since $\bar{p} : (A \otimes B)/E \to A \otimes B''$, $a \otimes b + E \mapsto a \otimes pb$,

Clearly, $\bar{d}$ is the inverse of $\bar{p}$.

$\therefore$ the sequence is exact at $A \otimes B$

3) $id \otimes p$ is surjective: Let $\sum a_i \otimes b''_i \in A \otimes B''$.

p surjection $\Rightarrow \forall i \, \exists b_i \in B : p(b_i) = b''_i$.

Thus

$$(id \otimes p)\left(\sum a_i \otimes b_i\right) = \sum a_i \otimes p(b_i) = \sum a_i \otimes b''_i .$$

$\square$

<u>Example 11.3.</u> Let G be a torsion group. Consider the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{p} \mathbb{Q}/\mathbb{Z} \to 0 .$$

Exercise 4 a on p. 105 $\Rightarrow \mathbb{Z} \otimes G \cong G$.

Exercise 3 on p. 100 $\Rightarrow \mathbb{Q} \otimes G \cong 0$.

Thus there is no injection $\mathbb{Z} \otimes G \to \mathbb{Q} \otimes G$.
In particular, $i \otimes id$ is not injective.

Corollary 11.4.

1) Let G be an abelian group. Let $m > 0$. Then

$$(\mathbb{Z}/m\mathbb{Z}) \otimes G \cong G/mG.$$

2) Let $m, n \in \mathbb{Z}$, $m, n > 0$, $\gcd(m, n) = d$. Then

$$(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}.$$

proof.

1) Consider the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/m\mathbb{Z} \to 0,$$

where $m : \mathbb{Z} \to \mathbb{Z}$, $x \mapsto mx$. Apply the functor $- \otimes G$ to the sequence. By Theorem 11.2, there is an exact sequence

$$\mathbb{Z} \otimes G \xrightarrow{m \otimes id} \mathbb{Z} \otimes G \xrightarrow{p \otimes id} (\mathbb{Z}/m\mathbb{Z}) \otimes G \to 0$$

Exercise 3 on p. 105 $\Rightarrow$ $m \otimes id$ is multiplication by m. Exercise 4 a on p. 105 $\Rightarrow$ $\mathbb{Z} \otimes G \cong G$. Thus we obtain an exact sequence

$$G \xrightarrow{m} G \xrightarrow{p \otimes id} (\mathbb{Z}/m\mathbb{Z}) \otimes G \to 0.$$

Here, $\ker(p \otimes id) = \operatorname{im}(m) = mG$,

and $\operatorname{im}(p \otimes id) = (\mathbb{Z}/m\mathbb{Z}) \otimes G$.

Thus

$$G/mG = G/\ker(p \otimes id) \cong \operatorname{im}(p \otimes id) = (\mathbb{Z}/m\mathbb{Z}) \otimes G.$$

2) $G = \mathbb{Z}/n\mathbb{Z}$

$$\Rightarrow (\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/m\mathbb{Z}) \otimes G$$

$$\cong G/mG$$

$$= (\mathbb{Z}/n\mathbb{Z})/m(\mathbb{Z}/n\mathbb{Z})$$

check this $\longrightarrow$

$$\cong \mathbb{Z}/d\mathbb{Z}. \quad \square$$

__Definition 11.5.__ Let A be an abelian group. Choose an exact sequence

$$0 \to R \xhookrightarrow{i} F \xrightarrow{p} A \to 0, \qquad (*)$$

where F is a free abelian group. Then also R is a free abelian group. Let B be an abelian group. Then there is an exact sequence

$$R \otimes B \xrightarrow{i \otimes id} F \otimes B \xrightarrow{p \otimes id} A \otimes B \to 0$$

Define $\mathrm{Tor}(A,B) = \ker(i \otimes id)$.

It can be shown that $\mathrm{Tor}(A,B)$ is independent of the choice of the exact sequence $(*)$.

Delete A from $(*)$. Then

$$C_*: \; 0 \to R \to F \to 0$$

is a chain complex, and also

$$C_* \otimes B: \; 0 \to R \otimes B \to F \otimes B \to 0$$

is a chain complex. Then $\mathrm{Tor}(A,B) = H_1(C_* \otimes B)$.

$\mathrm{Tor}(\ ,B)$ is a covariant functor $\mathcal{Ab} \to \mathcal{Ab}$:

Let A, A' be abelian groups, and let $\phi: A \to A'$ be a homomorphism. Let F, F' be abelian groups that give the exact sequence for A and A':

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, dashed, "f_1"] & F \arrow[r] \arrow[d, dashed, "f_0"] & A \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & R' \arrow[r] & F' \arrow[r] & A' \arrow[r] & 0
\end{tikzcd}$$

By Theorem 9.1, there are homomorphisms f_0 and f_1 that make the diagram commute. Tensor with B:

$$\begin{tikzcd}
C_* \otimes B: & 0 \arrow[r] & R \otimes B \arrow[r] \arrow[d, "f_1 \otimes id"] & F \otimes B \arrow[r] \arrow[d, "f_0 \otimes id"] & 0 \\
C'_* \otimes B: & 0 \arrow[r] & R' \otimes B \arrow[r] & F' \otimes B \arrow[r] & 0
\end{tikzcd}$$

Here $f_1 \otimes id$ and $f_0 \otimes id$ form a chain map. Thus they induce homomorphisms between homology groups. We define

$$\phi_* : \mathrm{Tor}(A,B) \to \mathrm{Tor}(A',B)$$

to be the homomorphism

$$H_1(C_* \otimes B) \to H_1(C'_* \otimes B)$$

induced by the chain map.

Basic properties of Tor

Let B be an abelian group. Then

$$\mathrm{Tor}(\ ,B) : \mathrm{Ab} \longrightarrow \mathrm{Ab}$$

is an additive covariant functor satisfying the following:

Tor 1: Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence of abelian groups. Then there is an exact sequence

$$0 \to \mathrm{Tor}(A',B) \to \mathrm{Tor}(A,B) \to \mathrm{Tor}(A'',B) \to A'\otimes B \to A\otimes B \to A''\otimes B \to 0$$

Tor 2: If A is torsion-free, then $\mathrm{Tor}(A,B)=0$.

Tor 3: $\mathrm{Tor}(\sum A_j, B) \cong \sum \mathrm{Tor}(A_j,B)$

$\mathrm{Tor}(A, \sum B_j) \cong \sum \mathrm{Tor}(A,B_j)$

Tor 4: $\mathrm{Tor}(\mathbb{Z}/m\mathbb{Z}, B) \cong B[m] = \{ b\in B \mid mb = 0\}$

Tor 5: $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$ for all A and B.

The proofs of these properties can be found in books on homological algebra.

Theorem 11.6. (Universal Coefficients Theorem for Homology)

1) Let X be a topological space, and let G be an abelian group. For every $n \geq 0$, there is an exact sequence

$$0 \to H_n(X) \otimes G \xrightarrow{\alpha} H_n(X;G) \to \mathrm{Tor}(H_{n-1}(X), G) \to 0, \quad (*)$$

where $\alpha([z] \otimes g) = [z \otimes g]$.

2) The sequence $(*)$ splits:

$$H_n(X;G) \cong (H_n(X) \otimes G) \oplus \mathrm{Tor}(H_{n-1}(X), G).$$

Notice: If $\mathrm{Tor}(H_{n-1}(X), G) = 0$, then $\alpha : H_n(X) \otimes G \to H_n(X;G)$ is an isomorphism.

proof. 1) Let (C_*, ∂) be a free chain complex. We show that, for every $n \geq 0$, there is an exact sequence

$$0 \to H_n(C_*) \otimes G \xrightarrow{\alpha} H_n(C_* \otimes G) \to \mathrm{Tor}(H_{n-1}(C_*), G) \to 0,$$

$$\alpha([z] \otimes g) = [z \otimes g].$$

The claim then follows by choosing $C_* = S_*(X)$.

Let $\begin{cases} Z_n = \text{the } n\text{-cycles} & \text{of } C_n. \\ B_n = \text{the } n \text{ boundaries} \end{cases}$

There is an exact sequence

$$0 \to Z_n \xrightarrow{i_n} C_n \xrightarrow{d_n} B_{n-1} \to 0, \quad (1)$$

where i_n = the inclusion, and $d_n = \partial_n| : C_n \to B_{n-1}$, ($\partial_n : C_n \to C_{n-1}$, $\partial_n(C_n) = B_{n-1}$).

Then there is a commutative diagram

$$\begin{tikzcd} C_n \arrow[rr, "\partial_n"] \arrow[dr, "d_n"'] & & C_{n-1} \\ & B_{n-1} \arrow[ur, "j_{n-1}"'] & \end{tikzcd}$$

j_{n-1} = the inclusion

Here C_{n-1} = free abelian $\Rightarrow$ B_{n-1} free abelian
Corollary 8.3 $\Rightarrow$ (1) is split exact. Then also (check this),

$$0 \to Z_n \otimes G \xrightarrow{i_n \otimes id} C_n \otimes G \xrightarrow{d_n \otimes id} B_{n-1} \otimes G \to 0 \qquad (2)$$

is split exact. Let

Z_* = subcomplex of C_*, n^{th} term = Z_n
B_*^+ = chain complex, n^{th} term = B_{n-1},

differentiation in Z_* : $\partial_n | : Z_n \to Z_{n-1}$, zero differential

$\Rightarrow$ differentials in $Z_* \otimes G$ are zero.

differentiation in B_*^+ : zero differentials

$\Rightarrow$ differentials in $B_*^+ \otimes G$ are zero.

The exact sequences in (2) give an exact sequence of chain complexes:

$$0 \to Z_* \otimes G \xrightarrow{i \otimes id} C_* \otimes G \xrightarrow{d \otimes id} B_*^+ \otimes G \to 0.$$

Thus there is a long exact sequence in homology:

$$\cdots \to H_{n+1}(B_*^+ \otimes G) \xrightarrow{\Delta_{n+1}} H_n(Z_* \otimes G) \xrightarrow{(i \otimes id)_*} H_n(C_* \otimes G)$$
$$\xrightarrow{(d \otimes id)_*} H_n(B_*^+ \otimes G) \xrightarrow{\Delta_n} H_{n-1}(Z_* \otimes G) \to \cdots, \qquad (3)$$

where Δ_n = the connecting homomorphism.

$Z_* \otimes G$ and $B^+_* \otimes G$ have zero differentials

$\Rightarrow \quad H_n(Z_* \otimes G) = (Z_* \otimes G)_n = Z_n \otimes G$,

and

$H_n(B^+_* \otimes G) = (B^+_* \otimes G)_n = B_{n-1} \otimes G.$

Thus the sequence (3) becomes

$$\cdots \to B_n \otimes G \xrightarrow{\Delta_{n+1}} Z_n \otimes G \xrightarrow{(i\otimes id)_*} H_n(C_* \otimes G) \xrightarrow{(d\otimes id)_*} B_{n-1} \otimes G \xrightarrow{\Delta_n} Z_{n-1} \otimes G \to \cdots \qquad (4)$$

Thus, for every n, there is an exact sequence

$$0 \to (Z_n \otimes G)/\operatorname{im}\Delta_{n+1} \xrightarrow{\alpha} H_n(C_* \otimes G) \xrightarrow{(d\otimes id)_*} \ker \Delta_n \to 0 \qquad (5)$$

Here α is induced by $(i \otimes id)_*$:

$$\alpha(z \otimes g + \operatorname{im}\Delta_{n+1}) = (i \otimes id)_*(z \otimes g) = [z \otimes g].$$

The diagram for the connecting homomorphism Δ_n is

$$\begin{array}{ccccccc} & & C_n \otimes G & \xrightarrow{d\otimes id} & B_{n-1} \otimes G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial \otimes id} & & & & \\ 0 \longrightarrow Z_{n-1} \otimes G & \xrightarrow{i \otimes id} & C_{n-1} \otimes G & & & & \end{array}$$

Let $b_{n-1} \otimes g \in B_{n-1} \otimes G$. Then

$$\Delta_n(b_{n-1} \otimes g) = (i \otimes id)^{-1}(\partial \otimes id)(d \otimes id)^{-1}(b_{n-1} \otimes g)$$

$$= b_{n-1} \otimes g \in Z_{n-1} \otimes G.$$

Thus $\Delta_n = j_{n-1} \otimes id$, where $j_{n-1}: B_{n-1} \hookrightarrow Z_{n-1}$ is the inclusion.

Thus the sequence (5) can be written as

$$0 \to (Z_n \otimes G)/\mathrm{im}(j_n \otimes id) \xrightarrow{\alpha} H_n(C_* \otimes G) \to \ker(j_{n-1} \otimes id)$$
$$\to 0 \qquad (6)$$

By the definition of homology, there is an exact sequence for every n:

$$0 \to B_{n-1} \xrightarrow{j_{n-1}} Z_{n-1} \to H_{n-1}(C_*) \to 0.$$

Tor1 $\Rightarrow$ there is an exact sequence

$$0 \to \mathrm{Tor}(B_{n-1}, G) \to \mathrm{Tor}(Z_{n-1}, G) \to \mathrm{Tor}(H_{n-1}(C_*), G)$$
$$\xrightarrow{\beta} B_{n-1} \otimes G \xrightarrow{j_{n-1} \otimes id} Z_{n-1} \otimes G \to H_{n-1}(C_*) \otimes G \to 0.$$

Z_{n-1} subgroup of C_{n-1} $\Rightarrow$ Z_{n-1} is torsion free

$$\overset{\text{Tor2}}{\Rightarrow} \quad \mathrm{Tor}(Z_{n-1}, G) = 0.$$

Thus

$$\mathrm{Tor}(H_{n-1}(C_*), G) \overset{\beta \text{ injective}}{\cong} \mathrm{im}(\beta) = \ker(j_{n-1} \otimes id).$$

Replace $n-1$ by n $\Rightarrow$

$$(Z_n \otimes G)/\mathrm{im}(j_n \otimes id) = \mathrm{coker}(j_n \otimes id) \cong H_n(C_*) \otimes G. \qquad (7)$$

The exact sequence (6) can now be rewritten as

$$0 \to H_n(C_*) \otimes G \xrightarrow{\alpha} H_n(C_* \otimes G) \to \mathrm{Tor}(H_{n-1}(C_*), G) \to 0. \quad (8)$$

2) It remains to show that the exact sequence (6) (or (8)) is split. There are inclusions

$$\mathrm{im}(\partial_{n+1} \otimes id) \subset Z_n \otimes G \subset \ker(\partial_n \otimes id) \subset C_n \otimes G.$$

Since the exact sequence

$$0 \to Z_n \otimes G \xrightarrow{i_n \otimes id} C_n \otimes G \xrightarrow{\partial_n \otimes id} B_{n-1} \otimes G \to 0$$

splits, it follows that $Z_n \otimes G$ is a direct summand of $C_n \otimes G$. Thus $Z_n \otimes G$ is a direct summand of $\ker(\partial_n \otimes id)$. Thus $(Z_n \otimes G)/\mathrm{im}(\partial_{n+1} \otimes id)$ is a direct summand of $\ker(\partial_n \otimes id)/\mathrm{im}(\partial_{n+1} \otimes id) = H_n(C_* \otimes G)$.
Here, $\mathrm{im}(\partial_{n+1} \otimes id) = \mathrm{im}(j_n \otimes id)$. Thus

$$(Z_n \otimes G)/\mathrm{im}(\partial_{n+1} \otimes id) \cong H_n(C_*) \otimes G,$$

by (7). Thus $H_n(C_*) \otimes G \cong$ direct summand of $H_n(C_* \otimes G)$

Therefore, $(*)$ splits. $\square$

Example

1) $G = \mathbb{Z}$: Ordinary singular homology can be considered as homology with coefficients in $\mathbb{Z}$:
Apply $- \otimes \mathbb{Z}$ to a singular chain complex $S_*(X)$.
Then $S_n(X) \otimes \mathbb{Z} = S_n(X)$ $\forall n$.

2) $G = \mathbb{R}, \mathbb{C}, \mathbb{Q} \Rightarrow G$ is torsion free

$\mathrm{Tor}\ 2 \Rightarrow \mathrm{Tor}(H_{n-1}(X), G) = 0 \quad \forall n$.

$\Rightarrow H_n(X; G) \cong H_n(X) \otimes G$.

$H_*(X; \mathbb{Q})$: rational homology

$H_*(X; \mathbb{R})$: real homology

$H_*(X; \mathbb{C})$: complex homology