__Theorem 8.13.__ Let A_* and B_* be free chain complexes, and let $f: A_* \to B_*$ be a chain map. Then f is a chain equivalence if and only if

$$f_{*n}: H_n(A_*) \to H_n(B_*)$$

is an isomorphism for every n.

__proof.__ Since chain homotopic maps induce the same homomorphisms in homology, it follows that if f is a chain equivalence, then $f_{*n}: H_n(A_*) \to H_n(B_*)$ is an isomorphism for every n. $(*)$

Assume then that ~~each~~ f_{*n} is an isomorphism for every n. The sequence

$$\cdots \to H_{n+1}(C(f)) \to H_n(A_*) \xrightarrow{f_{n*}} H_n(B_*) \to H_n(C(f)) \to \cdots$$

is exact by Lemma 8.12. Thus f_{n*} isomorphism $\Rightarrow$ $H_n(C(f)) = 0$ for all n. Thus $C(f)$ is acyclic. Theorem 8.11 $\Rightarrow$ f is a chain equivalence.

$\square$

$(*)$: This was proved in the course "Introduction to algebraic topology", see Homework 9, Exercise 5a.

Mapping cone

Let X be a topological space. The <u>cone</u> of X is

$$CX = (X \times I)/(X \times \{0\}).$$

Let $p: X \times I \to CX$ be the quotient map. The image $p(X \times \{0\})$ is called the <u>vertex</u> of X. The space X is identified with the image $p(X \times \{1\})$ in CX.

Let then X and Y be topological spaces and let $f: X \to Y$ be continuous. The <u>mapping cone</u> Cf of $f: X \to Y$ is the adjunction space

$$CX \amalg_f Y = (CX \amalg Y)/\sim,$$

where $\sim$ is the equivalence relation on $CX \amalg Y$, generated by $\{((x,1), f(x)) \mid x \in X\}$. Then Y is embedded as a closed subset of Cf.

The mapping cone Cf of $f: X \to Y$ is related to the (chain complex) mapping cone of

$$f_*: (S_*(X), \partial) \to (S_*(Y), \partial').$$

We may return to this later.

<u>Proposition 8.14.</u> Let X, Y be topological spaces, and let $f, g: X \to Y$ be continuous maps. If f and g are homotopic, then Cf and Cg are homotopy equivalent.

<u>Proof.</u> In many text books, see for example "Greenberg & Harper", Proposition 21.21. □

9. Acyclic Models

The method of acyclic models is a technique of constructing chain maps and chain homotopies. The <u>acyclic models theorem</u> is a theorem that is used to prove that two homology theories are isomorphic. The theorem was proved by Samuel Eilenberg and Saunders MacLane.

<u>Theorem 9.1.</u> i) Consider the following commutative diagram of abelian groups and homomorphisms:

$$\begin{tikzcd} F \arrow[r, "t"] \arrow[d, dashed, "c"] & G \arrow[r, "s"] \arrow[d, "b"] & G'' \arrow[d, "a"] \\ E' \arrow[r, "r"'] & E \arrow[r, "p"'] & E'' \end{tikzcd}$$

Assume the following:

1) The bottom row is exact.
2) $st = 0$
3) F is free abelian.

Then there is a homomorphism $c : F \to E'$ making the first square commute.

ii) Consider the following diagram of abelian groups and homomorphisms:

$$\begin{tikzcd} \cdots \arrow[r] & F_2 \arrow[r, "d_2"] \arrow[d, dashed, "t_2"'] & F_1 \arrow[r, "d_1"] \arrow[d, dashed, "t_1"'] & F_0 \arrow[r, "e"] \arrow[d, dashed, "t_0"'] & \operatorname{coker} d_1 \arrow[r] \arrow[d, "f"'] & 0 \\ \cdots \arrow[r] & E_2 \arrow[r, "\partial_2"'] & E_1 \arrow[r, "\partial_1"'] & E_0 \arrow[r, "\varepsilon"'] & \operatorname{coker} \partial_1 \arrow[r] & 0 \end{tikzcd}$$

(For abelian groups A, A' and a homomorphism $f: A \to A'$, the cokernel of f is $\operatorname{coker} f = A'/\operatorname{im} f$.)

Assume the following:

1) The rows are chain complexes.
2) Each F_i is free abelian.
3) The bottom row is exact. (Then the bottom row is acyclic.)

Then there exists a chain map $t: F \to E$ with $fe = \varepsilon t_0$.

<u>Proof.</u>

i) We show that $\operatorname{im} bt \subset \operatorname{im} r$: Since the bottom row is exact, $\operatorname{im} r = \ker p$. $\Rightarrow$ It suffices to show that $pbt = 0$. But $pbt = a\underbrace{st}_{0} = 0$.

$\therefore \operatorname{im} bt \subset \operatorname{im} r$

Thus:

$$\begin{tikzcd} & F \arrow[dl, dashed, "c"'] \arrow[d, "bt"] & \\ E' \arrow[r, "r"'] & \operatorname{im} r \arrow[r, "p"'] & 0 \end{tikzcd}$$

Theorem 8.2 $\Rightarrow$ $\exists$ homomorphism $c: F \to E'$ with $rc = bt$.

ii) We construct the t_i by induction on i.

$i = 0$:

$$\begin{tikzcd} & F_0 \arrow[dl, "t_0"'] \arrow[d, "fe"] & \\ E_0 \arrow[r, "\varepsilon"'] & \operatorname{coker} \partial_1 \arrow[r] & 0 \end{tikzcd}$$

F_0 free abelian, ε surjection: Theorem 8.2 $\Rightarrow$ $\exists$ homomorphism $t_0: F_0 \to E_0$, $\varepsilon t_0 = fe$.

Assume we have constructed t_n. Then t_{n+1} can be constructed by using part i. $\square$

<u>Definition 9.2.</u> If there is a diagram like in Theorem 9.1. ii, then we say that the chain map t is <u>over</u> f.

<u>Definition 9.3.</u> Let $F: \mathrm{Ab} \to \mathrm{Ab}$ be a functor. If, for all homomorphisms $f, g: A \to B$,

$$F(f+g) = F(f) + F(g),$$

then F is called <u>additive.</u>

<u>Definition 9.4.</u> Let $\mathcal{C}$ be a category and let $\mathcal{M}$ be a subset of obj $\mathcal{C}$. Then the ordered pair $(\mathcal{C}, \mathcal{M})$ is called a <u>category with models</u> $\mathcal{M}$. If $F: \mathcal{C} \to \mathrm{Ab}$ is a functor, then an <u>F-model set</u> is an indexed set $X = \{x_j \in F.M_j \mid j \in J\}$, where $\mathcal{M} = \{M_j \mid j \in J\}$ is an indexed family of models.

Let $C \in \mathcal{C}$, and let $\sigma: M_j \to C$ be a morphism in $\mathcal{C}$. Then $F\sigma: FM_j \to FC$ is a morphism in Ab, i.e., a group homomorphism. Let $x_j \in X$, say $x_j \in FM_j$. Then $(F\sigma)(x_j) \in FC$.

<u>Definition 9.5.</u> Let $\mathcal{C}$ be a category with models $\mathcal{M}$. Let $F: \mathcal{C} \to \mathrm{Ab}$ be a functor. We say that F is <u>free with base in $\mathcal{M}$</u>, if the following conditions hold:

i) For every object C of $\mathcal{C}$, the group FC is free abelian.

ii) There exists an F-model set $X = \{x_j \in FM_j \mid j \in J\}$ with the property that, for every object C of $\mathcal{C}$, the set $\{(F\sigma)(x_j) \mid x_j \in X, \sigma: M_j \to C\}$ is a basis of FC.

Example 9.6. Let $k \geq 0$. Let Δ^k be the standard k-simplex. Let $\mathcal{M} = \{\Delta^k\}$, i.e., $\mathcal{M}$ contains just one element Δ^k. Consider the category Top with models $\mathcal{M}$. Let

$$S_k : \mathrm{Top} \to \mathrm{Ab}, \quad X \mapsto S_k(X),$$

i.e. S_k assigns to a topological space X the k^{th} term of the singular chain complex $S_*(X)$ of X. Let

$$\delta : \Delta^k \to \Delta^k$$

be the identity map. Then $\delta \in S_k(\Delta^k)$, and $\mathcal{X} = \{\delta\}$ is an S_k-model set. Now, for every topological space X, $S_k(X)$ is a free abelian group, with basis all k-simplexes $\sigma : \Delta^k \to X$. For $\sigma : \Delta^k \to X$ we have

$$S_k(\sigma) : S_k(\Delta^k) \to S_k(X), \quad \sum m_i \sigma_i \mapsto \sum m_i \underbrace{(\sigma \circ \sigma_i)}_{\sigma_\#(\sigma_i)},$$

where $m_i \in \mathbb{Z}$.

Thus $S_k(\sigma)(\delta) = \sigma_\#(\delta) = \sigma \circ \delta = \sigma$. Hence the set

$$\{ S_k(\sigma)(\delta) \mid \delta \in \mathcal{X}, \ \sigma : \Delta^k \to X \}$$

is a basis for $S_k(X)$. Thus S_k is free with base in $\mathcal{M} = \{\Delta^k\}$.

Lemma 9.7. Let $\mathcal{C}$ be a category with models $\mathcal{M}$. Let $F : \mathcal{C} \to \mathrm{Ab}$ be a free functor with base $\mathcal{X} = \{x_j \in FM_j \mid j \in J\}$. Let $G : \mathcal{C} \to \mathrm{Ab}$ be a functor and let $\mathcal{Y} = \{y_j \in GM_j \mid j \in J\}$ be a G-model set. Then there exists a unique natural transformation $\tau : F \to G$ with $\tau_{M_j}(x_j) = y_j$ for every $j \in J$.

__proof.__ First, let's assume that τ exists, and let's prove uniqueness. Let $C \in \mathrm{obj}\,\mathcal{C}$, and let $\sigma : M_j \to C$ be a morphism. Since τ is a natural transformation, there is a commutative diagram.

$$\begin{tikzcd} FM_j \arrow[r, "F\sigma"] \arrow[d, "\tau_{M_j} = \tau_j"'] & FC \arrow[d, "\tau_C"] \\ GM_j \arrow[r, "G\sigma"'] & GC \end{tikzcd}$$

(Here τ_j denotes τ_{M_j}.)

Let $x_j \in X$. Then

$$\tau_C((F\sigma)(x_j)) = (G\sigma)(\tau_j(x_j)) = (G\sigma)(y_j).$$

The family

$$\{(F\sigma)(x_j) \mid x_j \in X,\ \sigma : M_j \to C\}$$

forms a basis of FC. It follows that τ_C is uniquely determined. Consequently, $\tau = \{\tau_C\}$ is unique.

We then construct τ. Define $\tau_C : FC \to GC$ as follows: $\tau_C((F\sigma)(x_j)) = (G\sigma)(y_j)$, then extend by linearity (FC free abelian with basis $(F\sigma)(x_j)$).

__Check:__ These τ_C form a natural transformation $F \to G$. Thus we should check that for any morphism $\phi : C \to D$ the following diagram commutes:

$$\begin{tikzcd} FC \arrow[r, "F\phi"] \arrow[d, "\tau_C"'] & FD \arrow[d, "\tau_D"] \\ GC \arrow[r, "G\phi"'] & GD \end{tikzcd}$$

FC free abelian $\Rightarrow$ it suffices to evaluate $\tau_D \circ (F\varphi)$ and $(G\varphi)\circ \tau_C$ on a typical basis element $(F\sigma)(x_j)$. Then:

$$\begin{aligned}(G\varphi)\circ\tau_C : (F\sigma)(x_j) \longmapsto (G\varphi)\big(\tau_C((F\sigma)(x_j))\big) &= (G\varphi)\big[(G\sigma)(y_j)\big] \\ &= (G\varphi\circ G\sigma)(y_j) \underset{\uparrow \ G \text{ functor}}{=} (G(\varphi\sigma))(y_j)\end{aligned}$$

and

$$\begin{aligned}\tau_D\circ(F\varphi) : (F\sigma)(x_j) \longmapsto \tau_D\big[(F\varphi)((F\sigma)(x_j))\big] &= \tau_D\big[(F\varphi\circ F\sigma)(x_j)\big] \\ \underset{\uparrow \ F \text{ functor}}{=} \tau_D\big[(F(\varphi\sigma)(x_j))\big] &= G(\varphi\sigma)(y_j).\end{aligned}$$

$\square$

<u>Lemma 9.8.</u> Let $\mathcal{C}$ be a category with models $\mathcal{M}$. Consider the following commutative diagram of functors $\mathcal{C} \to Ab$ and natural transformations

$$\begin{tikzcd} F \arrow[r, "\tau"] \arrow[d, dashed, "\gamma"'] & G \arrow[r, "\sigma"] \arrow[d, "\beta"] & G'' \arrow[d, "\alpha"] \\ E' \arrow[r, "\rho"'] & E \arrow[r, "\pi"'] & E'' \end{tikzcd} \qquad (*)$$

Assume the following:

1) $\sigma\tau = 0$ (this means that $\sigma_C\tau_C = 0$ for every object C)
2) $\operatorname{im}\rho = \underbrace{\ker \pi}_{\text{on } \mathcal{M}}$ (this means that $\operatorname{im}\rho_M = \ker \pi_M$ for every model $M \in \mathcal{M}$)
3) F is free with basis in $\mathcal{M}$.

Then there exists a natural transformation $\gamma : F \to E'$ making the first square commute.

proof. By 3), there is an F-model set $\{x_j \in FM_j \mid j \in J\}$ that is a base for F. For every j, there is a commutative diagram in Ab:

$$\begin{tikzcd}
FM_j \arrow[r, "\gamma_j"] \arrow[d, dashed] & GM_j \arrow[r, "\sigma_j"] \arrow[d, "\beta_j"] & G''M_j \arrow[d, "\alpha_j"] \\
E'M_j \arrow[r, "\rho_j"'] & EM_j \arrow[r, "\pi_j"'] & E''M_j
\end{tikzcd}$$

This diagram satisfies the hypotheses of Theorem 9.1 i). Now, every $x_j \in FM_j$ determines some $y'_j \in E'M_j$. The y'_j form an E'-model set. Lemma 9.7 $\Rightarrow$ there is a natural transformation $\psi : F \to E'$ satisfying $\psi_j(x_j) = y'_j$ for every j. It remains to check that the first square in $(*)$ commutes. Setting $y_j = \rho_j(y'_j)$ defines an E-model set. Then $\beta\gamma, \rho\psi : F \to E$ are natural transformations, $(\rho\psi)_j(x_j) = y_j = (\beta\gamma)_j(x_j)$. Uniqueness (Lemma 9.7) $\Rightarrow \beta\gamma = \rho\psi$. $\square$

Let **Comp** be the category whose objects are chain complexes and whose morphisms are chain maps.

Let $\mathcal{C}$ be a category and let $E : \mathcal{C} \to \mathbf{Comp}$ be a functor. Let $C \in \mathcal{C}$. Then $E(C)$ is a chain complex, and hence $H_n(E(C))$ is defined for every n. If $E_i = 0$ for all $i < 0$, we call E nonnegative. (Here $E_i(C)$ denotes $(E(C))_i$.) Assume E is nonnegative. Then the complex EC may be lengthened:

$$\cdots \to E_2C \to E_1C \to E_0C \to H_0(EC) \to 0,$$

since $H_0(EC) = \operatorname{coker}(E_1C \to E_0C)$. For every $k \geq 0$, E determines a functor

$$E_k : \mathcal{C} \to Ab, \quad C \mapsto E_kC.$$

Definition 9.9. Let $E : \mathcal{C} \to \mathrm{Comp}$ be a functor. Let C be an object in $\mathcal{C}$. If $H_n(EC) = 0$ for all $n > 0$, we call C *E-acyclic.*

Theorem 9.10. (Acyclic Models)

Let $\mathcal{C}$ be a category with models $\mathcal{M}$. Let $F, E : \mathcal{C} \to \mathrm{Comp}$ be two nonnegative functors. For every $k \geq 0$, assume the following:

1) F_k is free with base in $\mathcal{M}_k \subset \mathcal{M}$.
2) Each model M in $\mathcal{M}$ is E-acyclic.

Then:

i) For every natural transformation $\varphi : H_0 F \to H_0 E$, there is a natural chain map $\tau : F \to E$ over φ. In other words, there is a commutative diagram

$$\begin{tikzcd}
{} \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] \arrow[d, "\tau_1"] & F_0 \arrow[r] \arrow[d, "\tau_0"] & H_0F \arrow[r] \arrow[d, "\varphi"] & 0 \\
{} \arrow[r, "\partial_2"'] & E_1 \arrow[r, "\partial_1"'] & E_0 \arrow[r] & H_0E \arrow[r] & 0
\end{tikzcd}$$

ii) If $\tau, \tau' : F \to E$ are natural chain maps over φ, then τ and τ' are naturally chain homotopic.

iii) Assume that E_k is free with base $\mathcal{M}_k \subset \mathcal{M}$, and assume that each model M in $\mathcal{M}$ is F-acyclic. If φ is a natural equivalence, then every natural chain map $\tau : F \to E$ over φ is a natural chain equivalence.

Proof. 1) We should show that for every natural transformation $\varphi: H_0F \to H_0E$, there is a sequence of natural transformations $\tau_k: F_k \to E_k$, $k \geq 0$, s.t. the diagram

$$\begin{tikzcd}
\arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] \arrow[d, "\tau_1"] & F_0 \arrow[r] \arrow[d, "\tau_0"] & H_0F \arrow[r] \arrow[d, "\varphi"] & 0 \\
\arrow[r, "d_2"'] & E_1 \arrow[r, "d_1"'] & E_0 \arrow[r] & H_0E \arrow[r] & 0
\end{tikzcd}$$

commutes. Since $H_0(EC) = \operatorname{coker}(E_1C \to E_0C)$, it follows that the bottom row is exact at E_0C, for every $C \in \operatorname{obj} \mathcal{C}$. The proof is by induction on $k \geq 0$:

$k=0$: We have the diagram:

$$\begin{tikzcd}
F_0 \arrow[r, "d_0"] & H_0F \arrow[r, "0"] \arrow[d, "\varphi"] & 0 \arrow[d, "0"] \\
E_0 \arrow[r] & H_0E \arrow[r] & 0
\end{tikzcd}$$

This diagram satisfies the conditions of Lemma 9.8. Thus Lemma 9.8 $\Rightarrow$ there is a natural transformation $\tau_0: F_0 \to E_0$ making the first square commute.

$k>0$: Again, use Lemma 9.8 to construct τ_{k+1} when τ_k has been constructed.

2) Assume next that $\tau, \tau': F \to E$ both are natural chain maps over φ. We should find natural transformations $s_k: F_k \to E_{k+1}$, $k \geq -1$, with

$$\partial_{k+1} s_k + s_{k-1} d_k = \tau_k - \tau'_k.$$

$$\begin{tikzcd}
{} \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] \arrow[d, shift right, "\gamma_1"'] \arrow[d, shift left, "\gamma_1'"] \arrow[dl] & F_0 \arrow[r] \arrow[d, shift right, "\gamma_0"'] \arrow[d, shift left, "\gamma_0'"] \arrow[dl, "s_0"'] & H_0F \arrow[r] \arrow[d, "\varphi"] \arrow[dl, "s_{-1}"'] & 0 \arrow[d, "0"] \arrow[dl] \\
{} \arrow[r, "\partial_2"'] & E_1 \arrow[r, "\partial_1"'] & E_0 \arrow[r] & H_0E \arrow[r] & 0 .
\end{tikzcd}$$

Define $s_{-1} = 0$. Define the s_k, $k \geq 0$, by induction:
Write $\theta_k = \gamma_k - \gamma_k'$. Consider the diagram

$$\begin{tikzcd}
F_0 \arrow[r, "id"] & F_0 \arrow[r] \arrow[d, "\theta_0"] & 0 \arrow[d, "0"] \\
E_1 \arrow[r, "\partial_1"'] & E_0 \arrow[r] & H_0E
\end{tikzcd}$$

γ_0, γ_0' both over φ $\Rightarrow$ the square on the right commutes.
The bottom row is exact $\Rightarrow$ Lemma 9.8 applies
$\Rightarrow$ $\exists$ natural transformation $s_0 : F_0 \to E_1$ making the first square commute. Thus $\partial_1 s_0 = \theta_0 = \gamma_0 - \gamma_0'$.

$k > 0$: Consider the diagram

$$\begin{tikzcd}
F_k \arrow[r, "id"] & F_k \arrow[r] \arrow[d, "\theta_k - s_{k-1} d_k"] & 0 \arrow[d, "0"] \\
E_{k+1} \arrow[r, "\partial_{k+1}"'] & E_k \arrow[r, "\partial_k"'] & E_{k-1}
\end{tikzcd}$$

By assumption, the bottom row is exact for every model M. Lemma 9.8 $\Rightarrow$ $\exists$ natural transformation $s_k : F_k \to E_{k+1}$ with $\partial_{k+1} s_k = \theta_k - s_{k-1} d_k$, assuming the square on the right commutes. Check that it commutes:

$$
\begin{aligned}
\partial_k(\Theta_k - s_{k-1}d_k) &= \partial_k\Theta_k - (\partial_k s_{k-1})d_k \\
&= \partial_k\Theta_k - (\Theta_{k-1} - s_{k-2}d_{k-1})d_k \quad \leftarrow \text{by induction} \\
&= \partial_k\Theta_k - \Theta_{k-1}d_k \quad (d_{k-1}d_k = 0) \\
&= 0 \quad (\Theta \text{ is a chain map}).
\end{aligned}
$$

3) Assume next that each E_k is free with base $\mathcal{M}_k \subset \mathcal{M}$ and that each model M in $\mathcal{M}$ is F-acyclic. Assume φ is a natural equivalence. Then φ has the inverse $\varphi^{-1}: Ho(E) \to Ho(F)$. 1) $\Rightarrow \exists$ natural chain map $\sigma: E \to F$ over φ^{-1}. Then $\sigma\tau: F \to F$ is a natural chain map over $\varphi^{-1}\varphi = id$, where $id: Ho(F) \to Ho(F)$. Clearly, $id_F: F \to F$ is a natural chain map over $id: Ho(F) \to Ho(F)$. Then 2) $\Rightarrow \exists$ natural chain homotopy $\sigma\tau \simeq id_F$. Similarly, $\tau\sigma \simeq id_E$. Then $\tau: F \to E$ is a natural chain equivalence.

$\square$

<u>Definition 9.11.</u> Let (S_*, ∂) be a nonnegative chain complex. An <u>augmentation</u> of (S_*, ∂) is a surjective homomorphism $\varepsilon: S_0 \to \mathbb{Z}$ with $\varepsilon\partial_1 = 0$. A chain map $f: S_* \to S_*$ is called <u>augmentation preserving</u> if there is a commutative diagram

$$
\begin{array}{ccc}
S_0 & \xrightarrow{\varepsilon} & \mathbb{Z} \\
{\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle id} \\
S_0 & \xrightarrow[\varepsilon']{} & \mathbb{Z}
\end{array}
$$

Corollary 9.12. Let $\mathcal{C}$ be a category with models $\mathcal{M}$. Let F, E be functors from $\mathcal{C}$ to the category of augmented chain complexes.

1) Assume that each F_k is free with base in $\mathcal{M}$ and each model M is totally E-acyclic (this means: $\tilde{H}_n(EM) = 0$ for all $n \geq 0$). Then there exist natural chain maps $F \to E$ that are augmentation preserving, and any two such nat. chain maps are naturally chain homotopic.

2) Assume that, for all $k \geq 0$, both F_k and E_k are free with bases in $\mathcal{M}$, and that every model is both totally E-acyclic and totally F-acyclic. Then every augmentation preserving natural chain map is a natural chain equivalence.

proof.

1) In the proof of Theorem 9.10, replace $F_0 \to H_0F \to 0$ and $E_0 \to H_0F \to 0$ by their respective augmentations. Then we get

$$\begin{tikzcd}
\cdots \arrow[r] & F_1 \arrow[r, "d_1"] \arrow[d, dashed, "\tau_1"] & F_0 \arrow[r, "\varepsilon"] \arrow[d, dashed, "\tau_0"] & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0 \arrow[d] \\
\cdots \arrow[r] & E_1 \arrow[r, "\partial_1"'] & E_0 \arrow[r, "\varepsilon'"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Lemma 9.8 $\Rightarrow$ get τ_0
By induction: get τ_i, $i > 0$. $\Big\}$ Then τ is a natural augm. preserving chain map

If there are two such augm. preserving chain maps, they can be shown to be naturally chain homotopic as in the proof of Theorem 9.10.

2) Assume both F and E are free and acyclic. Then there are augmentation preserving natural chain maps $\tau: F \to E$ and $\sigma: E \to F$. The identity chain map $id: F \to F$ is also augmentation preserving. Uniqueness $\Rightarrow$ there is a natural chain homotopy $\sigma\tau \simeq id_F$ and a natural chain homotopy $\tau\sigma \simeq id_E$. Then τ and σ are natural chain equivalences.

$\square$

10. Tensor products

Definition 10.1. Let A and B be abelian groups. The *tensor product* $A \otimes B$ of A and B is the abelian group having the following presentation:

Generators: all ordered pairs $(a,b) \in A \times B$.

Relations: $(a+a', b) = (a,b) + (a',b)$ and
$(a, b+b') = (a,b) + (a,b')$, for all $a, a' \in A$ and $b, b' \in B$.

Let F be the free abeliangroup generated by all relations $(a,b) \in A \times B$. Let N be the subgroup of F generated by all relations, i.e., by elements of the form $(a+a', b) - (a,b) - (a', b)$ or of the form $(a, b+b') - (a,b) - (a,b')$. Then $A \otimes B = F/N$. Denote the coset $(a,b)+N$ by $a \otimes b$. Then a typical element in $A \otimes B$ has an expression of the form $\sum m_i (a_i \otimes b_i)$ for $m_i \in \mathbb{Z}$.