Claim. $Z_k(W_*(S^n)) = \langle \gamma_k \rangle$, for $1 \le k \le n$.

Assume $r, s \in \mathbb{Z}$ and $d_k(r A_{k,*}(\beta_k) + s\beta_k) = 0$.
Then

$$\begin{aligned} 0 &= d_k(r A_{k,*}(\beta_k) + s\beta_k) \\ &= r d_k A_{k,*}(\beta_k) + s d_k(\beta_k) \\ &= r(-1)^k d_k(\beta_k) + s d_k(\beta_k) \\ &= [r(-1)^k + s] d_k(\beta_k). \end{aligned}$$

$W_{k-1}(S^n)$ is a free abelian group $\Rightarrow$ either
$r(-1)^k + s = 0$ or $d_k(\beta_k) = 0$.

1. $r(-1)^k + s = 0$: Then $s = (-1)^{k+1} r$ $\Rightarrow$ $r A_{k,*}(\beta_k) + s\beta_k$
$= r[A_{k,*}(\beta_k) + (-1)^{k+1}\beta_k] = r\gamma_k$.
$\Rightarrow d_k(r A_{k,*}(\beta_k) + s\beta_k) \in Z_k(W_*(S^n))$
$\hookrightarrow \in \langle \gamma_k \rangle$.

$\therefore \langle \gamma_k \rangle = Z_k(W_*(S^n))$.

2. Show: $d_k(\beta_k) = 0$ can not happen

Assume $d_k(\beta_k) = 0$. Then also $d_k A_{k,*}(\beta_k) = (-1)^k d_k(\beta_k)$
$= 0$.

Thus $B_{k-1}(W_*(S^n)) = \operatorname{im} d_k = 0$, since β^n and $A_{k,*}(\beta^n)$ generate $W_k(S^n)$.

Assume $n > k-1 > 0$. Now $\langle \gamma_{k-1}\rangle \subset Z_{k-1}(W_*(S^n))$
$\Rightarrow Z_{k-1}(W_*(S^n)) \neq 0$.
If $B_{k-1}(W_*(S^n)) = 0$, then $0 \neq H_{k-1}(W_*(S^n)) \cong H_{k-1}(S^n)$.
Contradiction.

Assume $n > k-1$, $k=1$. Then $Z_0(W_*(S^n)) = W_0(S^n)$ is a free abelian group of rank 2 (S^n has 2 0-cells). Thus $H_0(W_*(S^n)) = Z_0(W_*(S^n))/\underbrace{B_0(W_*(S^n))}_{0}$ is a free abelian of rank 2.
However,
$$H_0(W_*(S^n)) \cong H_0(S^n) \cong \mathbb{Z}.$$
Contradiction.

Assume $1 < k \leq n$. Then

$$d_k(\beta^n) \in B_{k-1}(W_*(S^n)) = Z_{k-1}(W_*(S^n)) = \langle \gamma_{k-1}\rangle.$$

$H_{k-1}(S^n)=0$ By the case above, since $n > k-1$ (and see part iv on p. 69)

$\Rightarrow d_k(\beta^n) = m\gamma_{k-1}$ for some $m \in \mathbb{Z}$,

$$\Rightarrow d_k A_{k,*}(\beta^n) \underset{\uparrow \text{p.69}}{=} (-1)^k d_k \beta^n = (-1)^k m \gamma_{k-1}.$$

$\Rightarrow \operatorname{im} d_k \subset \langle m\gamma_{k-1}\rangle$

But $0 = H_{k-1}(S^n) = \dfrac{Z_{k-1}(W_*(S^n))}{B_{k-1}(W_*(S^n))}$

$\Rightarrow \operatorname{im} d_k = B_{k-1}(W_*(S^n)) = Z_{k-1}(W_*(S^n)) = \langle \gamma_{k-1}\rangle$

$\Rightarrow m = \pm 1$.

Thus

$$d_k(\beta_k) = \pm \gamma_{k-1} = \pm\left(A_{k-1,*}(\beta_{k-1}) + (-1)^k \beta_{k-1}\right)$$

as claimed.

$k=1$: Compute d_1 directly. Then β_1 = the upper half circle from 1 to -1, and $\beta_0 = 1$. Then

$$d_1(\beta_1) = i_*\partial_1(\beta_1) = [1] - [-1] \qquad ([\ ]: \text{homology class})$$

$\square$

Theorem 6.2. If n is odd, then

$$H_p(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & , \text{ if } p=0 \text{ or } p=n \\ \mathbb{Z}/2\mathbb{Z} & , \text{ if } p \text{ is odd and } 0<p<n \\ 0 & , \text{ otherwise} \end{cases}$$

If n is even, then

$$H_p(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & , \text{ if } p=0 \\ \mathbb{Z}/2\mathbb{Z} & , \text{ if } p \text{ is odd and } 0<p<n \\ 0 & , \text{ otherwise} \end{cases}$$

proof. Let $p: S^n \to S^n/\sim = \mathbb{R}P^n$ be the quotient map and let $D_k: W_k(\mathbb{R}P^n) \to W_{k-1}(\mathbb{R}P^n)$ denote the differentiation in $W_*(\mathbb{R}P^n)$. Since p is cellular, it induces a chain map $p_*: W_*(S^n) \to W_*(\mathbb{R}P^n)$. The diagram

$$\begin{tikzcd} W_k(S^n) \arrow[r, "d_k"] \arrow[d, "p_*"'] & W_{k-1}(S^n) \arrow[d, "p_*"] \\ W_k(\mathbb{R}P^n) \arrow[r, "D_k"'] & W_{k-1}(\mathbb{R}P^n) \end{tikzcd}$$

commutes.

Then

$$\begin{aligned}
D_k p_*(\beta_k) &= p_* d_k(\beta_k) \\
&= \pm p_*\left(A_{k-1,*}(\beta_{k-1}) + (-1)^k \beta_{k-1}\right) \\
&= \pm \Big[\underbrace{p_*\left(A_{k-1,*}(\beta_{k-1})\right)}_{= p_*(\beta_{k-1}),\ \text{see p. 67}} + p_*\left((-1)^k \beta_{k-1}\right) \Big] \\
&= \pm \left[1 + (-1)^k \right] p_*(\beta_{k-1}).
\end{aligned}$$

Here $p_*(\beta_k)$ generates $W_k(\mathbb{R}P^n)$. Thus:

k odd: $1+(-1)^k = 0 \Rightarrow D_k$ is the zero map

k even: $1+(-1)^k = 2 \Rightarrow D_k$ is multiplication by 2.

Abbreviate $W_k(\mathbb{R}P^n)$ to W_k. The cellular complex of $\mathbb{R}P^n$ is

$$\begin{tikzcd}
0 \arrow[r] & W_n \arrow[r, "D_n"] \arrow[d, phantom, "\cong" rotate=90] & W_{n-1} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \cdots \arrow[r] & W_4 \arrow[r, "D_4"] \arrow[d, phantom, "\cong" rotate=90] & W_3 \arrow[r, "D_3"] \arrow[d, phantom, "\cong" rotate=90] & W_2 \arrow[r, "D_2"] \arrow[d, phantom, "\cong" rotate=90] & W_1 \arrow[r, "D_1"] \arrow[d, phantom, "\cong" rotate=90] & W_0 \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "(-1)^n+1"] & \mathbb{Z} \arrow[r] & \cdots \arrow[r] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ $H_0(\mathbb{R}P^n) = H_0(W_*) = \mathbb{Z}$

$H_1(\mathbb{R}P^n) = H_1(W_*) = \mathbb{Z}/2\mathbb{Z}$

$H_2(\mathbb{R}P^n) = H_2(W_*) = 0/0 = 0$

$H_3(\mathbb{R}P^n) = H_3(W_*) = \mathbb{Z}/2\mathbb{Z}$

$\vdots$

$$H_n(\mathbb{R}P^n) = \begin{cases} 0/0 = 0, & \text{if } n \text{ is even} \\ \mathbb{Z}/0 = \mathbb{Z}, & \text{if } n \text{ is odd}. \end{cases}$$

$\square$

Some properties of CW-complexes

Theorem 6.3. (Cellular Approximation Theorem)

Let X and Y be CW complexes. Let X' be a (possibly empty) CW subcomplex of X. Let $g: X \to Y$ be a continuous map. Assume the restriction $g|X'$ is a cellular map. Then there exists a cellular map $f: X \to Y$ such that $f|X' = g|X'$ and $f \simeq g$ rel X'. $\square$

For $n \geq 0$, consider $s_n = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$ as the base point of S^n.

Definition 6.4. Let (X, x_0) be a topological pointed space and let $n \geq 0$. Then the group

$$\pi_n(X, x_0) = [(S^n, s_n), (X, x_0)]$$

is called the n^{th} homotopy group of X ($n \geq 1$). Usually $\pi_n(X, x_0)$ is abbreviated to $\pi_n(X)$. If $n \geq 2$, $\pi_n(X)$ is called a higher homotopy group of X.

Here $[(S^n, s_n), (X, x_0)]$ denotes the set of homotopy (rel s_n) classes of maps $(S^n, s_n) \to (X, x_0)$. It can be shown that $\pi_n(X)$ is a group for all $n \geq 1$. $n \geq 2 \Rightarrow \pi_n(X)$ abelian.

Let X, Y be topological spaces and let $f: X \to Y$ be a continuous map. The map f is called an <u>n-equivalence</u> for $n \geq 1$ if f induces a one-to-one correspondence between the path components of X and Y and if for every $x \in X$, $f_*: \pi_q(X, x) \to \pi_q(Y, f(x))$ is an isomorphism for $0 < q < n$ and an epimorphism for $q = n$. A map $f: X \to Y$ is called a <u>weak homotopy equivalence</u> if it is an n-equivalence for all $n \geq 1$.

Theorem 6.5. A map between CW complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence. □

Theorem 6.6 (Whitehead Theorem)

Let X and Y be path connected pointed spaces and let $f : (X, x_0) \to (Y, y_0)$ be a continuous map. If there is $n \geq 1$ such that

$$f_\# : \pi_q(X, x_0) \to \pi_q(Y, y_0)$$

is an isomorphism for $q < n$ and an epimorphism for $q = n$, then

$$f_* : H_q(X, x_0) \to H_q(Y, y_0)$$

is an isomorphism for $q < n$ and an epimorphism for $q = n$. Conversely, if X and Y are simply connected and if f_* is an isomorphism for $q < n$ and an epimorphism for $q = n$, then $f_\#$ is an isomorphism for $q < n$ and an epimorphism for $q = n$.

7. Natural transformations

<u>Definition 7.1.</u> Let $\mathcal{C}$ and $\mathcal{A}$ be categories. Let $F, G : \mathcal{C} \to \mathcal{A}$ be (covariant) functors. A <u>natural transformation</u> $\tau : F \to G$ is a one-parameter family of morphisms

$$\tau = \{ \tau_C : F(C) \to G(C) \mid C \in \mathrm{obj}\,\mathcal{C} \}$$

such that the following diagram commutes for every morphism $f : C \to C'$:

$$\begin{tikzcd}
F(C) \arrow[r, "Ff"] \arrow[d, "\tau_C"'] & F(C') \arrow[d, "\tau_{C'}"] \\
G(C) \arrow[r, "Gf"'] & G(C')
\end{tikzcd}$$

If F and G are both contravariant, the definition is similar, just the horizontal arrows should be reversed.

<u>Definition 7.2.</u> A natural transformation $\tau : F \to G$ is called a <u>natural equivalence</u> if τ_C is an equivalence for every $C \in \mathrm{obj}\,\mathcal{C}$. The functors F and G are called <u>isomorphic</u> (or <u>naturally equivalent</u>) if there is a natural equivalence between them.

<u>Recall:</u> An equivalence in a category $\mathcal{C}$ is a morphism $f : A \to B$, where $A, B \in \mathrm{obj}\,\mathcal{C}$, such that there exists a morphism $g : B \to A$ with $f \circ g = id_B$ and $g \circ f = id_A$.

<u>Example</u> Let $*$ be a one-point space, for example $* = \{a\}$. A function $h : * \to X$, where X is a set, is completely determined by its only value $x = h(a) \in X$. Denote h by h_x. Often x and h_x are identified (singular 0-simplexes in a space X are identified with the points in X). Let Sets denote the category whose objects are sets and whose morphisms are all functions between sets. For a set A, let $\mathrm{Hom}(*, A)$ denote the family of functions $* \to A$. Then $\mathrm{Hom}(*, A)$ is also a set (in fact, by the definition of a category, $\mathrm{Hom}(A, B)$ is always required to be a set...) Then

$$\mathrm{Hom}(*, \ ) : \text{Sets} \to \text{Sets}, \quad A \mapsto \mathrm{Hom}(*, A),$$

is a functor. To each map $f : A \to B$, A, B sets, $\mathrm{Hom}(*, \ )$ assigns the map (morphism)

$$\mathrm{Hom}(*, f) : \mathrm{Hom}(*, A) \to \mathrm{Hom}(*, B), \quad h \mapsto f \circ h.$$

For a set X, define

$$\mathcal{I}_X : X \to \mathrm{Hom}(*, X), \quad \mathcal{I}_X(x) = h_x.$$

Let $f : X \to Y$ be a map. Then the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathcal{I}_X"'] & Y \arrow[d, "\mathcal{I}_Y"] \\
\mathrm{Hom}(*, X) \arrow[r, "\mathrm{Hom}(*, f)"'] & \mathrm{Hom}(*, Y)
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\
h_x \arrow[r, mapsto] & h_{f(x)} = f \circ h_x
\end{tikzcd}$$

It follows that τ is a natural transformation. Moreover, every τ_X is an equivalence. Thus it follows that τ is a natural equivalence.

<u>Example</u> Similarly one can show that the identity functor on the category of abelian groups, denoted by $\mathcal{A}b$, is isomorphic to $\mathrm{Hom}(\mathbb{Z}, \ )$. Every homomorphism $f: \mathbb{Z} \to G$, when G is an abelian group, is completely determined by $f(1) \in G$.

8. Chain Equivalence

<u>Definition 8.1.</u> A chain complex C_* is called free if each of its terms is a free abelian group.

<u>Theorem 8.2.</u> Let F be a free abelian group, and let B and C be abelian groups. Let $g: B \to C$ be a surjective homomorphism. Let $h: F \to C$ be a homomorphism. Then there exists a homomorphism $f: F \to B$ with $g \circ f = h$.

$$\begin{tikzcd} & F \arrow[dl, dashed, "f"'] \arrow[d, "h"] & \\ B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

<u>Proof.</u> Let X be a basis for F. Since g is surjective, it follows that for every $x \in X$ there is $b_x \in B$ with $g(b_x) = h(x)$. The function $x \mapsto b_x$ defines a homomorphism $f: F \to B$ (extend $x \mapsto b_x$ by linearity). For every $x \in X$,

$$g(\varphi(x)) = g(b_x) = h(x).$$

Since $(g\circ\varphi)(x) = h(x)$ for the generators $x \in X$, it follows that $g\circ\varphi = h$. $\square$

<u>Corollary 8.3.</u> Let F be a free abelian group, and let B be an abelian group. Let $g : B \to F$ be a surjective homomorphism. Then $B = \ker g \oplus F'$, where $F' \cong F$.

<u>proof.</u> Consider the diagram

$$\begin{tikzcd}
 & F \arrow[d, "id"] \arrow[dl, dashed, "\varphi"'] & \\
B \arrow[r, "g"'] & F \arrow[r] & 0
\end{tikzcd}$$

By Theorem 8.2, there is a homomorphism $\varphi : F \to B$ with $g\circ\varphi = id$. Thus φ is injective. Check that $B = \ker g \oplus \operatorname{im}\varphi$: Let $b \in B$. Then

$$b = (\underbrace{b - \varphi(g(b))}_{\in F}) + \varphi(g(b)).$$

Here $\varphi(g(b)) \in \operatorname{im}\varphi$, and $g(b - \varphi(g(b))) = g(b) - \overbrace{(g\circ\varphi)}^{id}(g(b)) = 0$. Thus $b - \varphi(g(b)) \in \ker g$.

Assume $b \in \ker g \cap \operatorname{im}\varphi$. Then $b = \varphi(x)$, for some $x \in F$, and

$$0 = g(b) = g(\varphi(x)) = (g\circ\varphi)(x) = x.$$

Thus $b = \varphi(x) = \varphi(0) = 0$. It follows that $\ker g \cap \operatorname{im}\varphi = \{0\}$. Thus

$$B = \ker g \oplus \operatorname{im}\varphi,$$

where $F' = \operatorname{im}\varphi \cong F$. $\square$

Definition 8.4. Let A, B, C be abelian groups. An exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is called <u>split</u> (or a <u>split exact sequence</u>), if there is a homomorphism $s: C \to B$ with $ps = \mathrm{id}_C$.

Lemma 8.5. The following statements are equivalent:

1) The exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is split.
2) There is a homomorphism $q: B \to A$ with $qi = \mathrm{id}_A$.
3) A is a direct summand of B (that is, there exists a subgroup C' of B such that the restriction $p|: C' \to C$ is an isomorphism and $B = \mathrm{im}(i) \oplus C'$.

proof. See "Introduction to Algebraic Topology", Fall 2016, Homework 11, Exercise 1. $\square$

Theorem 8.6. Let F be a free abelian group. Let H be a subgroup of F. Then H is free abelian and $\mathrm{rank}(H) \leq \mathrm{rank}(F)$.

In particular: If F is finitely generated, then also H is finitely generated.

proof. We give two proofs. The first one only works when F has finite rank.

1) Assume F has finite rank n. We prove the claim by induction on n.

$n=1$: $F \cong \mathbb{Z}$. Any subgroup of $\mathbb{Z}$ is either 0 or $\cong \mathbb{Z}$. Thus H is free abelian and $\operatorname{rank}(H) \leq 1 = \operatorname{rank}(F)$.

Assume then F has a basis $\{x_1, \dots, x_n\}$. Set $F_n = \langle x_1, \dots, x_{n-1} \rangle$, and let $H_n = H \cap F_n$. By induction, H_n is a free abelian group, and $\operatorname{rank}(H_n) \leq n-1$. Now

$$H/H_n = H/(H \cap F_n) \cong (H+F_n)/F_n \subset F/F_n \cong \mathbb{Z}.$$

Thus H/H_n is isomorphic to a subgroup of $\mathbb{Z}$. If $H/H_n = 0$, then $H = H_n$. If $H/H_n \cong \mathbb{Z}$, then Corollary 8.3 $\Rightarrow$ $H = H_n \oplus \langle h \rangle$, where $\langle h \rangle \cong \mathbb{Z}$. Thus H is a free abelian group of rank $\leq (n-1)+1 = n$.

2) Proof in the case where $\operatorname{rank}(F)$ does not need to be finite: Let $\{x_k \mid k \in K\}$ be a basis of F. We assume $\{x_k \mid k \in K\}$ is well ordered. (That every nonempty set can be well ordered is equivalent to the axiom of choice.) For $k \in K$, let

$$F_k = \sum_{j<k} \langle x_j \rangle \quad , \quad \bar{F}_k = \sum_{j \leq k} \langle x_j \rangle$$

$$H_k = H \cap F_k \quad , \quad \bar{H}_k = H \cap \bar{F}_k .$$

Now,

$$F = \bigcup_k \bar{F}_k \quad , \quad H = \bigcup_k \bar{H}_k$$

and

$$H_k = H \cap F_k = H \cap F_k \cap \bar{F}_k = \bar{H}_k \cap F_k .$$

Thus

$$\bar{H}_k/H_k = \bar{H}_k/(\bar{H}_k \cap F_k) \cong (\bar{H}_k + F_k)/F_k \subset \bar{F}_k/F_k \cong \mathbb{Z}.$$

As in case 1, either $\bar{H}_k = H_k$, or $\bar{H}_k = H_k \oplus \langle h_k \rangle$, where $\langle h_k \rangle \cong \mathbb{Z}$.

Claim: H is a free abelian group with basis $\{h_k\}$.

[Then it will follow that $\operatorname{rank}(H) = \operatorname{card}\{h_k\} \leq \operatorname{card}(K) = \operatorname{rank} F$.]

proof of the claim:

Let H^0 = the subgroup of H generated by the h_k. Now

$$F = \bigcup_k F_k \implies \text{each } h \in H \text{ lies in some } \bar{F}_k.$$

Let $\mu(h)$ = the least index k with $h \in \bar{F}_k$. Assume $H \neq H^0$. Consider the set

$$A = \{\mu(h) \mid h \in H,\ h \notin H^0\}.$$

K is well ordered $\implies$ A has a least such index j. Let $h' \in H$ be such that $\mu(h') = j$ and $h' \notin H^0$. Then $\mu(h') = j \implies h' \in H \cap \bar{F}_j$. Thus

$$h' = a + m h_j, \quad \text{where } a \in H_j,\ m \in \mathbb{Z}$$

(since $\bar{H}_j = H \cap \bar{F}_j = H_j \oplus \langle h_j \rangle$).

$\implies a = h' - m h_j \in H$, $a \notin H^0$, and $a \in H_j$.
$\implies \mu(a) < j$. Contradiction.

$\therefore H = H^0$.

It remains to show that linear combinations of the h_k are unique. It suffices to show that if

$$m_1 h_{k_1} + \dots + m_n h_{k_n} = 0, \quad k_1 < \dots < k_n,$$

then $m_1 = \dots = m_n = 0$. Assume $m_n \neq 0$.

Then

$$m_n h_{k_n} = -m_1 h_{k_1} - \dots - m_{n-1} h_{k_{n-1}} \in \langle h_{k_n} \rangle \cap H_{k_n} = 0,$$

which is a contradiction. Thus the linear combinations of the h_k are unique, and it follows that H is a free abelian group with basis $\{h_k\}$. $\square$

<u>Definition 8.7.</u> A chain complex (A_*, ∂) is called acyclic, if $H_n(A_*) = 0$, for every n.

<u>Theorem 8.8.</u> A free chain complex (A_*, ∂) is acyclic if and only if it has a contracting homotopy.

<u>Recall:</u> A contracting homotopy of a chain complex (A_*, ∂) is a sequence of homomorphisms $C = \{C_n : A_n \to A_{n+1}\}$ such that for all $n \in \mathbb{Z}$,

$$\partial_{n+1} C_n + C_{n-1} \partial_n = id_{A_n}.$$

Thus: A contracting homotopy is a chain homotopy between the identity map $\{id_{A_n}\}$ of A and the zero map on A_*.

<u>proof of Theorem 8.8.</u>

Assume (A_*, ∂) has a contracting homotopy. Then $id_* : A_* \to A_*$ and $0 : A_* \to A_*$ are chain homotopic. Thus they induce the same homomorphisms on homology:

$$id = H_n(id) = H_n(0) = 0 : H_n(A_*) \to H_n(A_*).$$

Then $H_*(A_*) = 0$, i.e., $H_n(A_*) = 0 \ \forall n$.
(This true without the assumption that (A_*, ∂) is free.)

Thus (A_*, ∂) is acyclic.

Assume then that $H_n(A_*) = 0 \; \forall \, n \geq 0$. Theorem 8.6 $\Rightarrow Z_n(A_*) \subset A_n$ is free abelian. Since $H_{n-1}(A_*) = 0$, it follows that

$$\partial_n(A_n) = B_{n-1}(A_*) = Z_{n-1}(A_*).$$

$$\begin{tikzcd}
 & Z_{n-1}(A_*) \subset A_{n-1} \arrow[dl, "s_{n-1}"'] \arrow[d, "id"] \\
A_n \arrow[r, "\partial_n"'] & Z_{n-1}(A_*) \subset A_{n-1}
\end{tikzcd}$$

$\leftarrow$ surjective

$\partial_n \leftarrow$ surjective

Theorem 8.2 $\Rightarrow \exists$ homomorphism $s_{n-1} : Z_{n-1}(A_*) \to A_n$ with $\partial_n s_{n-1} = id$. Then

$$id - s_{n-1}\partial_n : A_n \to A_n,$$

and for $a \in A_n$,

$$\partial_n(id - s_{n-1}\partial_n)(a) = \partial_n(a) - \underbrace{\partial_n s_{n-1}}_{id} \partial_n(a) = 0.$$

$\Rightarrow im(id - s_{n-1}\partial_n) \subset Z_n(A_*)$. Let

$$t_n = s_n(id - s_{n-1}\partial_n) : A_n \to A_{n+1}.$$

Then

$$\begin{aligned}
\partial_{n+1} t_n + t_{n-1}\partial_n &= \overbrace{\partial_{n+1}[s_n}^{id}(id - s_{n-1}\partial_n)] \\
&\quad + s_{n-1}(id - s_{n-2}\underbrace{\partial_{n-1})\partial_n}_{0} \\
&= id - s_{n-1}\partial_n + s_{n-1}\partial_n \\
&= id.
\end{aligned}$$

$\Rightarrow \{t_n\}$ is a contracting homotopy of A_*. $\square$

Definition 8.9. Let (A_*, ∂) and (B_*, ∂') be chain complexes and let $f: (A_*, \partial) \to (B_*, \partial')$ be a chain map. The <u>mapping cone</u> of f is the chain complex $C(f)$, where

1) the n^{th} term of $C(f)$ is $C(f)_n = A_{n-1} \oplus B_n$,

2) the differentiation $D_n : C(f)_n \to C(f)_{n-1}$ is given by

$$D(a_{n-1}, b_n) = (-\partial_{n-1} a_{n-1}, f_{n-1} a_{n-1} + \partial'_n b_n).$$

Using matrix notation,

$$D = \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix}, \quad \begin{pmatrix} -\partial_{n-1} & 0 \\ f_{n-1} & \partial'_n \end{pmatrix}\begin{pmatrix} a_{n-1} \\ b_n \end{pmatrix} = \begin{pmatrix} -\partial_{n-1} a_{n-1} \\ f_{n-1} a_{n-1} + \partial'_n b_n \end{pmatrix}.$$

Lemma 8.10. Let $f: A_* \to B_*$ be a chain map, where A_* and B_* are free chain complexes. Then $C(f)$ is a free chain complex.

proof:

$$D_{n-1} \circ D_n = \begin{pmatrix} -\partial_{n-2} & 0 \\ f_{n-2} & \partial'_{n-1} \end{pmatrix}\begin{pmatrix} -\partial_{n-1} & 0 \\ f_{n-1} & \partial'_n \end{pmatrix}$$

$$= \begin{pmatrix} \overbrace{(-\partial_{n-2})(-\partial_{n-1})}^{0} & 0 \\ \underbrace{f_{n-2}(-\partial_{n-1}) + \partial'_{n-1} f_{n-1}}_{0 \text{ since } f \text{ is a chain map}} & \underbrace{\partial'_{n-1}\partial'_n}_{0} \end{pmatrix}$$

$\therefore$ $C(f)$ is a chain complex

A_{n-1}, B_n free abelian $\Rightarrow C(f)_n = A_{n-1} \oplus B_n$ free abelian.

$\therefore$ $C(f)$ is free. $\square$

Theorem 8.11. Let $f: A_* \to B_*$ be a chain map between free chain complexes. Assume $C(f)$ is acyclic. Then f is a chain equivalence.

proof. Lemma 8.10 $\Rightarrow$ $C(f)$ is a free chain complex. Theorem 8.8 $\Rightarrow$ $C(f)$ has a contracting homotopy (since $C(f)$ is free and acyclic). Thus there are $T_n : C(f)_n \to C(f)_{n+1}$ such that

$$D_{n+1} T_n + T_{n-1} D_n = id : C(f)_n \to C(f)_n .$$

Exercise: Write T_n as a 2×2-matrix and use it to show that f is a chain equivalence.

$\square$

Lemma 8.12. Let (A_*, ∂) and (B_*, ∂') be chain complexes, and let $f: A_* \to B_*$ be a chain map. Then there is an exact sequence

$$\cdots \to H_{n+1}(C(f)) \to H_n(A_*) \xrightarrow{f_*} H_n(B_*) \to H_n(C(f)) \to \cdots$$

proof. Define a chain complex A^+_* as follows:

$$(A^+_*)_n = A_{n-1}, \quad \partial^+_n = \partial_{n-1}.$$

Let $p_n : C(f)_n = A_{n-1} \oplus B_n = (A^+_*)_n \oplus B_n \to (A^+_*)_n$, $(a,b) \mapsto a$

be the projection, and let

$$i_n : B_n \to (A^+_*)_n \oplus B_n = C(f)_n, \quad b \mapsto (0,b)$$

be the inclusion. Then there exists a short exact sequence of chain complexes:

$$0 \to B_* \xrightarrow{i} C(\phi) \xrightarrow{p} A^+_* \to 0.$$

Thus there is a long exact sequence in homology:

$$\cdots \to H_{n+1}(B_*) \to H_{n+1}(C(\phi)) \to H_{n+1}(A^+_*) \xrightarrow{\Delta} H_n(B_*)$$

$$\to H_n(C(\phi)) \to \cdots$$

where Δ is the connecting homomorphism.

Consider the following diagram:

$$\begin{tikzcd}
 & A_n \oplus B_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "D_{n+1}"] & A_n \arrow[r] & 0 \\
0 \arrow[r] & B_n \arrow[r, "i_n"] & A_{n-1} \oplus B_n &
\end{tikzcd}$$

Let $a \in A_n$ be a cycle, i.e, $\partial_n a = 0$.

Then $p_{n+1}(a, 0) = a$,

$D_{n+1}(a, 0) = (-\underbrace{\partial_n a}_{0}, \phi_n(a) + \underbrace{\partial'_{n+1}(0)}_{0}) = (0, \underbrace{\phi_n(a)}_{\in B_n})$

and

$i_n(\phi_n(a)) = (0, \phi_n(a))$.

Thus

$$\begin{aligned}
\Delta_{n+1} : H_{n+1}(A^+_*) = H_n(A_*) &\to H_n(B_*) \\
[a] &\mapsto [\phi_n(a)] \\
&= \phi_*[a],
\end{aligned}$$

where $\phi_* : H_n(A) \to H_n(B)$ is the homomorphism induced by the chain map ϕ.

Thus the connecting homomorphism Δ equals ϕ_*, which proves the claim. $\square$