proof. 1) Let E be a CW decomposition of X, and let $E' \subset E$ be a CW decomposition of Y. Let $k \geq 1$, and let M_k be a set that contains exactly one point from each k-cell in $E - E'$. Lemma 4.23 $\Rightarrow$ X_Y^{k-1} is a strong deformation retract of $X_Y^k - M_k$.

$H_p(X_Y^k - M_k, X_Y^{k-1}) = 0 \quad \forall\, p \geq 0$:

Let Y be a topol. space and let A be a def. retract of Y. Then $\exists$ continuous $r: Y \to A$: $r \circ i = id_A$, $i \circ r \simeq id_Y$, where $i: A \hookrightarrow Y$ is the inclusion. Long exact sequence:

$$\cdots \to H_p(A) \xrightarrow{i_*} H_p(Y) \xrightarrow{\alpha} H_p(Y,A) \xrightarrow{\beta} H_{p-1}(A) \xrightarrow{i_*} H_{p-1}(Y) \to \cdots$$

r_*

$$\to H_0(A) \xrightarrow{i_*} H_0(Y) \xrightarrow{\alpha} H_0(Y,A) \xrightarrow{\beta} 0$$

$$\left.\begin{array}{l} r_* \circ i_* = (r \circ i)_* = (id_A)_* = id \Rightarrow i_* \text{ inj} \\ i_* \circ r_* = (i \circ r)_* = (id_Y)_* = id \Rightarrow i_* \text{ surj} \end{array}\right\} \; i_* \text{ bij.}$$

$$\left.\begin{array}{l} i_* \text{ surj} \Rightarrow \ker \alpha = H_p(Y) \Rightarrow \operatorname{im} \alpha = 0 \\ i_* \text{ inj} \Rightarrow \operatorname{im} \beta = 0 \Rightarrow \ker \beta = H_p(Y,A) \end{array}\right\} \Rightarrow H_p(Y,A) = \ker\beta = \operatorname{im}\alpha = 0 \text{ for } p > 0$$

$p = 0$: $\ker\alpha = H_0(Y) \Rightarrow 0 = \operatorname{im}\alpha = \ker\beta = H_0(Y,A)$.

The long exact sequence of the triple $(X_Y^k, X_Y^k - M_k, X_Y^{k-1})$ has the portion

$$\to \underset{\substack{\cong \\ 0}}{H_p(X_Y^k - M_k, X_Y^{k-1})} \to H_p(X_Y^k, X_Y^{k-1}) \to H_p(X_Y^k, X_Y^k - M_k)$$

$$\to \underset{\substack{\cong \\ 0}}{H_{p-1}(X_Y^k - M_k, X_Y^{k-1})} \to$$

$\Rightarrow$ an isomorphism $H_p(X_Y^k, X_Y^{k-1}) \to H_p(X_Y^k, X_Y^k - M_k)$,

$\forall\, k, p \geq 1$.

Let $\overline{X_Y^{k-1}}$ = the closure of X_Y^{k-1} in X_Y^k

$(X_Y^k - M_k)^\circ$ = the interior of $X_Y^k - M_k$ in X_Y^k.

(in fact, $X_Y^k - M_k \Subset X_Y^k$)

Now, $X_Y^{k-1} \Subset X$, $X_Y^{k-1} \subset X_Y^k \Rightarrow \overline{X_Y^{k-1}} = X_Y^{k-1}$.

$\Rightarrow \overline{X_Y^{k-1}} = X_Y^{k-1} \subset X_Y^k - M_k = (X_Y^k - M_k)^\circ$.

Excision I $\Rightarrow$ the inclusion $(X_Y^k - X_Y^{k-1}, (X_Y^k - M_k) - X_Y^{k-1})$

$\hookrightarrow (X_Y^k, X_Y^k - M_k)$

induces isomorphisms

$$H_p(X_Y^k, X_Y^k - M_k) \cong H_p(X_Y^k - X_Y^{k-1}, (X_Y^k - \cancel{X_Y^{k-1}}) - M_k).$$

Now,

$$X_Y^k - X_Y^{k-1} = \coprod_x \{e_x^k \in E - E' : e_x^k \text{ is a } k\text{-cell}\}.$$

Then, for all $p, k \geq 1$,

$$H_p(X_Y^k - X_Y^{k-1}, (X_Y^k - X_Y^{k-1}) - M_k) \cong \sum_x H_p(e_x^k, e_x^k - M_k). \quad (*)$$

Since $e_x^k \cong \mathbb{R}^k$, it follows that

$$H_p(e_x^k, e_x^k - M_k) \cong H_p(\mathbb{R}^k, \mathbb{R}^k - \{0\}).$$

Then, for all $p, k \geq 1$,

$$\begin{aligned} H_p(X_Y^k, X_Y^{k-1}) &\cong H_p(X_Y^k, X_Y^k - M_k) \\ &\cong H_p(X_Y^k - X_Y^{k-1}, (X_Y^k - X_Y^{k-1}) - M_k) \\ &\cong \sum_x H_p(e_x^k, e_x^k - M_k) \\ &\cong \sum_x H_p(\mathbb{R}^k, \mathbb{R}^k - \{0\}). \qquad (**) \end{aligned}$$

The long exact sequence of the pair $(\mathbb{R}^k, \mathbb{R}^k - \{0\})$ is

$$\cdots \to H_{p+1}(\mathbb{R}^k - \{0\}) \to \underbrace{H_{p+1}(\mathbb{R}^k)}_{\cong\, 0} \to H_{p+1}(\mathbb{R}^k, \mathbb{R}^k - \{0\}) \to H_p(\mathbb{R}^k - \{0\})$$

$$\to \underbrace{H_p(\mathbb{R}^k)}_{\cong\, 0 \text{ FOR } p>0} \to \cdots \qquad (***)$$

Since $\mathbb{R}^k - \{0\} \simeq S^{k-1}$, $H_p(\mathbb{R}^k - \{0\}) = \mathbb{Z}$ if $p = 0, k-1$, and $H_p(\mathbb{R}^k - \{0\}) = 0$ otherwise. $(H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z})$

Since we are proving that the filtration of X by the X^k_γ is cellular, we want to show that $H_p(X^k_\gamma, X^{k-1}_\gamma) = 0$ $\forall\, p \neq k$. Therefore, we are only interested in the case $p \neq k$.

a) Assume $p \geq 1$, $p \neq k$. $(***)$ becomes

$$0 \to H_{p+1}(\mathbb{R}^k, \mathbb{R}^k - \{0\}) \to \underbrace{H_p(\mathbb{R}^k - \{0\})}_{\cong\, H_p(S^{k-1})} \to 0$$

Then $H_{p+1}(\mathbb{R}^k, \mathbb{R}^k - \{0\}) \cong H_p(S^{k-1}) \cong 0$ if $\underbrace{p \neq k-1}_{p+1 \neq k}$.

$\Rightarrow H_p(\mathbb{R}^k, \mathbb{R}^k - \{0\}) = 0$, for $p \geq 2$, $p \neq k$.

b) Assume $p = 1$, $k \neq 1$. Then

$$0 = H_1(\mathbb{R}^k) \to H_1(\mathbb{R}^k, \mathbb{R}^k - \{0\}) \to \tilde{H}_0(\mathbb{R}^k - \{0\}).$$

(Use reduced homology $\tilde{H}_\bullet$.) Then $\mathbb{R}^k - \{0\}$ path connected

$\Rightarrow \tilde{H}_0(\mathbb{R}^k - \{0\}) \cong 0 \Rightarrow H_1(\mathbb{R}^k, \mathbb{R}^k - \{0\}) = 0.$

c) Finally, assume $p=0$. Then

$$\to H_0(\mathbb{R}^n - \{0\}) \xrightarrow{\alpha} H_0(\mathbb{R}^n) \xrightarrow{\beta} H_0(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \to 0$$

Here α is an isomorphism$^{k>1}$. Then $\ker \beta = H_0(\mathbb{R}^n)$ and $\operatorname{im}\beta = 0$. $\Rightarrow H_0(\mathbb{R}^n, \mathbb{R}^n - \{0\}) = 0$.

Notice: α is an isomorphism, since $\mathbb{R}^n$ is path connected and $\mathbb{R}^n - \{0\} \neq \emptyset$. See the proof from last fall, where we calculated H_0 for a path connected space.
(also path connected) $k>1$

$\rightarrow k=1 \Rightarrow 2$ conn comps $\Rightarrow \alpha$ surjection o.k. (no need to be an isom.)

It follows from a, b, c, that $H_p(X^k_Y, X^{k-1}_Y) = 0$ for all $p \neq k$, $p \geq 0$.

Condition ii) in the definition of a cellular filtration:

$\forall\, m \geq 0$, $\forall$ continuous $\sigma: \Delta^m \to X$ $\exists\, n$ s.t. $\sigma(\Delta^m) \subset X^n_Y$.

Now, Δ^m compact $\Rightarrow \sigma(\Delta^m)$ compact $\overset{4.12}{\Rightarrow} \sigma(\Delta^m)$ is contained in a finite CW subcomplex of X
$\Rightarrow \sigma(\Delta^m) \subset X^k_Y$ for some k.

$\therefore$ The X^k_Y form a cellular filtration for X.

2) Theorem 5.6: X = cellular space, $k \geq 0$.
$\Rightarrow H_k(W_*(X)) \cong H_k(X, X^{-1})$.

Since $X^{-1}_Y = Y$, it follows that $H_k(W_*(X,Y)) \cong H_k(X,Y)$.

$\square$

Using the proof of Theorem 5.9 gives:

Theorem 5.10. Let (X, E) be a CW complex, and let (Y, E') be a CW subcomplex of (X, E). Then, for $k \geq 0$, $W_k(X, Y)$ is a free abelian group of rank r_k, where r_k = the number of k-cells in $E - E'$. In particular, $W_k(X) = W_k(X, \emptyset)$ is a free abelian group whose rank equals the number of k-cells in E.

Here: The rank of a free abelian group G means the cardinality of the set of generators of G.

proof. By $(**)$ on p. 55, $\forall\, p, k \geq 1$,

$$W_k(X, Y) = H_k(X^k_Y, X^{k-1}_Y) \cong \sum_\lambda H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\}),$$

where λ goes through all the k-cells of $\underbrace{X}_{\text{in } E-E'}$, i.e., over a set with cardinality r_k.

The proof of part a on p. 56 $\Rightarrow$

$$H_{p+1}(\mathbb{R}^k, \mathbb{R}^k - \{0\}) \cong H_p(S^{k-1}), \quad \forall\, p \geq 1,\ p \neq k$$

Thus, for $p = k-1 \geq 1$ $(\Rightarrow k > 1)$,

$$H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\}) \cong H_{k-1}(S^{k-1}) \cong \mathbb{Z}$$

Thus, for $k \geq 2$,

$$W_k(X, Y) \cong \sum_\lambda H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\}) \cong \sum_\lambda (\mathbb{Z})_\lambda .$$

$\therefore$ the claim is proved for $k \geq 2$.

$k=1$: Exact sequence

$$0 = H_1(\mathbb{R}) \to H_1(\mathbb{R}, \mathbb{R}(0)) \to \underbrace{\overbrace{\tilde{H}_0(\mathbb{R}-\{0\})}^{\cong S^0}}_{\cong \mathbb{Z}} \to \tilde{H}_0(\mathbb{R}) = 0$$

$\Rightarrow H_1(\mathbb{R}, \mathbb{R}-\{0\}) \cong \mathbb{Z}$ $\therefore$ the claim proved for $k=1$

$k=0$: For top. space X and for $A \subset X$,

$$H_0(X, A) \cong \sum_j H_0(X_j, A \cap X_j),$$

where $\{X_j \mid j \in J\}$ is the set of path components of X. Thus need to calculate

$$W_0(X, Y) = H_0(X^0_Y, Y),$$

where $X^0_Y =: X^{(0)} \cup Y$. Let $\{Y_i \mid i \in I\}$ be the set of path components of Y. Let

$$X^{(0)'}_i = \{e^0_\lambda \in X^{(0)} \mid e^0_\lambda \notin Y\}.$$

$X^{(0)}$ is discrete, $Y \in X \Rightarrow$ the path components of X^0_Y are the Y_i, $i \in I$, and the $\{e^0_\lambda\}$ where $e^0_\lambda \notin Y$. Therefore,

$$H_0(X^0_Y, Y) \cong \sum_\lambda H_0(\{e^0_\lambda\}, \overbrace{Y \cap \{e^0_\lambda\}}^{\phi}) \oplus \sum_i \underbrace{H_0(Y_i, Y \cap Y_i)}_{=0}$$

$$\cong \sum_\lambda H_0(\{e^0_\lambda\}, \underbrace{Y \cap \{e^0_\lambda\}}_{\phi})$$

$$\cong \sum_\lambda H_0(\{e^0_\lambda\}).$$

(Thm: X path conn, $\phi \neq A \subset X \Rightarrow H_0(X, A) = 0$)

$\Rightarrow W_0(X, Y) = H_0(X^0_Y, Y)$ is a free abelian group of rank r_0, where r_0 = the cardinality of $\{e^0_\lambda \in X^{(0)} \mid e^0_\lambda \notin Y\}$.

$\square$

Example

1) $\mathbb{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$

$$\Rightarrow W_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & \text{for } k = 0, 2, \dots, 2n \\ 0, & \text{otherwise} \end{cases}$$

$$\cdots \to 0 \to 0 \to 0 \to \underset{\substack{\uparrow \\ 2n}}{\mathbb{Z}} \to 0 \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \underset{\substack{\uparrow \\ 2}}{\mathbb{Z}} \to 0 \to \underset{\substack{\uparrow \\ 0}}{\mathbb{Z}} \to 0$$

Then

$$H_k(\mathbb{C}P^n) = H_k(W_*(\mathbb{C}P^n)) = \begin{cases} \mathbb{Z}, & \text{for } k = 0, \dots, 2n \ (k \text{ even}) \\ 0, & \text{otherwise} \end{cases}$$

2) $\mathbb{H}P^n = e^0 \cup e^4 \cup \cdots \cup e^{4n}$

$$\Rightarrow W_k(\mathbb{H}P^n) = \begin{cases} \mathbb{Z}, & \text{for } k = 0, 4, 8, \dots, 4n \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow H_k(\mathbb{H}P^n) = H_k(W_*(\mathbb{H}P^n)) = \begin{cases} \mathbb{Z}, & \text{for } k = 0, 4, 8, \dots, 4n \\ 0, & \text{otherwise} \end{cases}$$

3) $\mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n$

$$\Rightarrow W_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & \text{for } k = 0, \dots, n \\ 0, & \text{otherwise} \end{cases}$$

$$\cdots \to 0 \to 0 \to \underset{\substack{\uparrow \\ n}}{\mathbb{Z}} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \cdots \to \mathbb{Z} \to \underset{\substack{\uparrow \\ 0}}{\mathbb{Z}} \to 0$$

$H_k(\mathbb{R}P^n) = H_k(W_*(\mathbb{R}P^n))$ can not be calculated without knowing the differentials d_n.

Corollary 5.11. Let X be a compact CW complex of dimension m [the dimension of X = the maximal dimension of its (finitely many) cells]. Then:

1) $H_p(X)$ is finitely generated for every $p \geq 0$.

2) $H_p(X) = 0$ for all $p > m$.

3) $H_m(X)$ is free abelian. $\square$

Theorem 5.12. Let (X, E) be a CW complex and let (Y, E') be a CW subcomplex of (X, E). Let $p : X \to X/Y$ be the quotient map. Then, for every $k \geq 0$, p induces an isomorphism

$$p_* : H_k(X, Y) \xrightarrow{\cong} H_k(X/Y, *) \cong \tilde{H}_k(X/Y).$$

Proof. As earlier, consider X/Y as a CW complex with CW decomposition $E'' = (E - E') \cup \{*\}$. The quotient map p is a cellular map of pairs,

$$p : (X, Y) \to (X/Y, *),$$

and p maps the cells in $X - Y$ homeomorphically onto cells in $(X/Y) - \{*\}$. p cellular $\Rightarrow$ it induces a chain map

$$p_* : W_*(X, Y) \to W_*(X/Y; *).$$

The equation $(**)$ on p. 55 gives the isomorphism

$$W_k(X, Y) = H_k(X^k_Y, X^{k-1}_Y) \cong \underset{\substack{\uparrow \\ \text{over the } k\text{-cells} \\ \text{in } X - Y}}{\sum} H_k(e^k, e^k - M_k), \quad \text{for all } k \geq 1.$$

The quotient map p maps the cells in $E - E'$ homeomorphically onto their images $\Rightarrow p_{*,k}$ is an isomorphism for all $k \geq 1$.

$k=0$: $W_0(X,Y) = H_0(X^0_Y, \underbrace{X^{-1}_Y}_{Y})$.

Check: p induces a bijection from the family of path components of $X^0_Y = X^{(0)} \cup Y$ not containing Y to the family of path components of $(X/Y)^0$ not containing $*$.

$\Rightarrow p_*$ is an isomorphism of chain complexes.

$\Rightarrow p_*$ induces the isomorphisms

$$H_*(X,Y) \cong H_*(X/Y, *) \cong \tilde{H}_*(X/Y).$$

$(H_k(X,Y) = H_k(X, X^{-1}) \cong H_k(W_*(X,Y))$
$H_k(X/Y, *) = H_k(W_*(X/Y, *))$ $)$ ☐

Corollary 5.13. Let X_i be a CW complex with a CW subcomplex Y_i, for $i=1,2$. Let $f: (X_1, Y_1) \to (X_2, Y_2)$ be a continuous (not necessarily cellular) map that induces a homeomorphism $\bar{f}: X_1/Y_1 \to X_2/Y_2$. Then, for all $k \geq 0$, f induces an isomorphism

$$f_k: H_k(X_1, Y_1) \to H_k(X_2, Y_2).$$

Proof. Let $p_i: X_i \to X_i/Y_i$ be the quotient map, for $i=1,2$. The following diagram commutes:

$$\begin{array}{ccc} H_k(X_1, Y_1) & \xrightarrow{f_k} & H_k(X_2, Y_2) \\ \downarrow{\scriptstyle p_{1,k}} & & \downarrow{\scriptstyle p_{2,k}} \\ \tilde{H}_k(X_1/Y_1) & \xrightarrow[\tilde{f}_k]{} & \tilde{H}_k(X_2/Y_2) \end{array}$$

Now: $\bar{f}$ homeomorphism $\Rightarrow \tilde{f}_k$ an isomorphism

Thm 5.12 $\Rightarrow p_{1,k}$, $p_{2,k}$ isomorphisms.

$\Rightarrow f_k$ is also an isomorphism. $\square$

<u>Corollary 5.14</u> (Excision) Let X be a CW complex and let Y_1 and Y_2 be CW subcomplexes of X with $X = Y_1 \cup Y_2$. Then the inclusion
$Y_1 \cap Y_2 \neq \emptyset$

$$i : (Y_1, Y_1 \cap Y_2) \hookrightarrow (X, Y_2)$$

induces an isomorphism

$$i_k : H_k(Y_1, Y_1 \cap Y_2) \to H_k(X, Y_2), \quad \text{for all } k \geq 0.$$

<u>proof.</u> The inclusion i induces a homeomorphism

$$\bar{i} : Y_1/(Y_1 \cap Y_2) \longrightarrow (Y_1 \cup Y_2)/Y_2 = X/Y_2.$$

To see that $\bar{i}$ is a homeomorphism, consider the following commutative diagram. (Clearly, $\bar{i}$ is a bijection.)

$$\begin{tikzcd}
Y_1 \arrow[r, hook, "i"] \arrow[d, "p_1"'] & Y_1 \cup Y_2 \arrow[d, "p"] \\
Y_1/(Y_1 \cap Y_2) \arrow[r, "\bar{i}"'] & (Y_1 \cup Y_2)/Y_2
\end{tikzcd}$$

Since p, i are continuous, p_1 is a quotient map, it follows that $\bar{i}$ is continuous. Let $A \Subset Y_1 \cup Y_2$. Then $p^{-1}(pA) = A$ if $A \cap Y_2 = \emptyset$ and $p^{-1}(pA) = A \cup Y_2$ if $A \cap Y_2 \neq \emptyset$. Since $Y_2 \Subset Y_1 \cup Y_2$, it follows that in both cases $p^{-1}(pA) \Subset Y_1 \cup Y_2$. Then $p(A) \Subset Y_1 \cup Y_2/Y_2$ and p is a closed map. Also, i is a closed map. Let $F \Subset Y_1/(Y_1 \cap Y_2)$. Then $\bar{i}(F) = \bar{i}(p_1(p_1^{-1}F)) = (p \circ i)(p_1^{-1}F) \Subset (Y_1 \cup Y_2)/Y_2$ since $p^{-1}F \Subset Y_1$. Then $\bar{i}$ is a closed map. Consequently, $\bar{i}$ is a homeomorphism. Now, Corollary 5.13 $\Rightarrow$ the claim. $\square$

Corollary 5.15. (Mayer-Vietoris) Let X be a CW complex, let Y_1 and Y_2 be CW subcomplexes of X with $X = Y_1 \cup Y_2$. Then there is an exact sequence

$$\cdots \to H_k(Y_1 \cap Y_2) \to H_k(Y_1) \oplus H_k(Y_2) \to H_k(X) \to H_{k-1}(Y_1 \cap Y_2) \to \cdots$$

Proof. See how one obtains the M-V sequence by using excision... $\square$

Definition 5.16. Let (X, E) be a finite CW complex. Let α_i = the number of i-cells in E. The Euler-Poincaré characteristic of (X, E) is

$$\chi(X) = \sum_{i \geq 0} (-1)^i \alpha_i .$$

Remark 5.17. It can be shown that

$$\chi(X) = \sum_{i \geq 0} (-1)^i (\operatorname{rank} H_i(X)).$$

Thus $\chi(X)$ does not depend on the CW decomposition of X.

Example

Circle :  1 0-cell, 1 1-cell $\Rightarrow (-1)^0 + (-1)^1 = -1+1 = 0$

same

 2 0-cells, 2 1-cells $\Rightarrow (-1)^0 \cdot 2 + (-1)^1 \cdot 2 = -2+2 = 0$

2-sphere :  1 0 cell, 1 2-cell $\Rightarrow (-1)^0 + (-1)^2 = 2$

2 0-cells, 2 1-cells, 2 2-cells

same

$\Rightarrow (-1)^0 \cdot 2 + (-1)^1 \cdot 2 + (-1)^2 \cdot 2 = 2-2+2 = 2.$

Theorem 5.18. Let X and X' be finite CW complexes. Then

$$\chi(X \times X') = \chi(X)\chi(X').$$

Proof. Let E and E' be CW decompositions of X and X', respectively. Then $E'' = \{e \times e' \mid e \in E, e' \in E'\}$ is a CW decomposition of $X \times X'$. If $e \in E, e' \in E'$, e is an i-cell and e' is a j-cell, then $e \times e'$ is an $(i+j)$-cell. Let α_i be the number of i-cells in X, and let α'_j be the number of j-cells in X'. Then the number of k-cells of $X \times X'$ is

$$\beta_k = \sum_{i+j=k} \alpha_i \alpha'_j .$$

Now,
$$\left(\sum (-1)^i \alpha_i\right)\left(\sum (-1)^j \alpha'_j\right) = \sum_{i,j} (-1)^{i+j} \alpha_i \alpha'_j$$

$$= \sum_{k \geq 0} (-1)^k \left(\sum_{i+j=k} \alpha_i \alpha'_j\right) = \sum_{k \geq 0} (-1)^k \beta_k .$$

Then $\chi(X)\chi(X') = \chi(X \times X')$. $\square$

6. Homology of $\mathbb{R}P^n$

Consider S^n as a CW complex having two k-cells, for all $0 \leq k \leq n$. Let the k-cells be e^k_1, e^k_2. Here

$\bar{e}^k_1$ = the closed northern hemisphere of S^k
$\bar{e}^k_2$ = the closed southern hemisphere of S^k

The k-skeleton of S^n : the cells in S^k

Let $m_1^k \in e_1^k$, $m_2^k \in e_2^k$, assume m_1^k and m_2^k are antipodal points, i.e, that $m_2^k = -m_1^k$.
Then

$$W_k(S^n) = H_k(S^k, S^{k-1}) = H_k(e_1^k, e_1^k - \{m_1^k\}) \oplus H_k(e_2^k, e_2^k - \{m_2^k\}),$$

for $k \geq 1$ (see $(**)$ on p. 55).

The proof of Theorem 5.10 $\Rightarrow$ the groups

$$H_k(e_1^k, e_1^k - \{m_1^k\}) \text{ and } H_k(e_2^k, e_2^k - \{m_2^k\})$$

are infinite cyclic (i.e., $\cong \mathbb{Z}$). Denote the generator of $H_k(e_1^k, e_1^k - \{m_1^k\})$ by β_k.

(But sometimes this is denoted by β^k. Sorry!)

Let $a_k : S^k \to S^k$, $x \mapsto -x$,

be the antipodal map, and let

$$A_k : (S^k, S^{k-1}) \to (S^k, S^{k-1})$$

be the antipodal map of pairs. Then the restriction

$$A_k| : (e_1^k, e_1^k - \{m_1^k\}) \to (e_2^k, e_2^k - \{m_2^k\})$$

is a homeomorphism. $\Rightarrow A_{k*}(\beta_k)$ is a generator of $H_k(e_2^k, e_2^k - \{m_2^k\})$. Then $\{\beta_k, A_{k,*}(\beta_k)\}$ is a basis for $W_k(S^n)$. The diagram

$$\begin{tikzcd}
H_k(S^k, S^{k-1}) \arrow[r, "A_{k,*}"] \arrow[d, "\partial_*"] & H_k(S^k, S^{k-1}) \arrow[d, "\partial_*"] \\
H_{k-1}(S^{k-1}) \arrow[r, "a_{k-1,*}"'] & H_{k-1}(S^{k-1})
\end{tikzcd} \qquad (\Delta)$$

commutes. The vertical maps are connecting homomorphisms.

Here: $A_{k,*} : H_k(S^n, S^{n-1}) \to H_n(S^n, S^{n-1})$

$$\begin{array}{ccc} H_k(S^n,S^{n-1}) & \to & H_n(S^n,S^{n-1}) \\ \cong & & \cong \\ \mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} \\ (x,y) & \mapsto & (y,x) \end{array},$$

since $A_{k,*}$ interchanges β_k and $A_{k*}(\beta_k)$.

$$\begin{array}{cccc} a_{k-1,*} : & H_{k-1}(S^{n-1}) & \to & H_{n-1}(S^{n-1}) \\ & \cong \mathbb{Z} & & \cong\mathbb{Z} \\ & x & \mapsto & (-1)^k x \end{array}$$

because we have the following result:

If $n \geq 1$, then the antipodal map $a_n : S^n \to S^n$ has degree $(-1)^{n+1}$ [$n > 0$, $f : S^n \to S^n$ continuous. If $f_* : H_n(S^n) \to H_n(S^n)$ is multiplication by m, then we say that f has degree m.]

For a proof, see Rotman Thm 6.23.

Let $p : S^n \to S^n/\sim \; = \mathbb{R}P^n$, where $x \sim y$ if $x = y$ or $x = -y$.

Here $\mathbb{R}P^n = e_0 \cup e_1 \cup \dots \cup e_n$, where e_i = an i-cell, $0 \leq i \leq n$.

Then p is a cellular map $\Rightarrow$ it induces a chain map

$$p_* : W_*(S^n) \to W_*(\mathbb{R}P^n).$$

$\forall x \in S^k \subset S^n$: $p(A_k(x)) = p(-x) = p(x) \Rightarrow p \circ A_k = p$

$\Rightarrow p_* \circ A_{k,*} = p_* \Rightarrow p_*(\beta_k) = p_*(A_{k*}(\beta_k)) \; \forall k$,

$\mathbb{R}P^n$ has one k-cell $\Rightarrow W_k(\mathbb{R}P^n) \cong \mathbb{Z}$.

Then $p_*(\beta_k)$ generates $W_k(\mathbb{R}P^n)$.

Lemma 6.1. For all $k \geq 0$, $W_k(S^n)$ has a basis $\{\beta_k, A_{k,*}(\beta_k)\}$ having the following properties:

1) $p_*(\beta_k) = p_*(A_{k,*}(\beta_k))$, and $p_*(\beta_k)$ generates $W_k(\mathbb{RP}^n)$.

2) For $k > 0$, the differentiation $d_k : W_k(S^n) \to W_{k-1}(S^n)$ satisfies
$$d_k(\beta_k) = \pm\left(A_{k-1,*}(\beta^{k-1}) + (-1)^k \beta^{k-1}\right).$$

proof. 1) o.k.

2) The generator $\beta_k \in H_k(e^k_i, e^k_i - \{m^k_i\})$. By the equations on p. 55, we may identify β_k with its image in $H_k(S^k, S^{k-1}) = W_k(S^n)$.
The differential d_k is defined as the composite $i_* \partial_k$, where

$$\begin{tikzcd}
H_k(S^k, S^{k-1}) \arrow[rr, "d_k"] \arrow[dr, "\partial_k"'] & & H_{k-1}(S^{k-1}, S^{k-2}) \\
& H_{k-1}(S^{k-1}) \arrow[ur, "i_*"'] &
\end{tikzcd}$$

when ∂_k is the connecting homomorphism and i_* is induced by $i : (S^{k-1}, \emptyset) \hookrightarrow (S^{k-1}, S^{k-2})$.

By the definition of $A_{k,*}$ and $a_{k-1,*}$ and by (Δ),

$$d_k(A_{k,*}(\beta^k)) = i_* \partial_k A_{k,*}(\beta^k) \overset{\Delta}{=} i_* a_{k-1,*} \partial_k(\beta^k)$$
$$= (-1)^k i_* \partial_k(\beta^k) = (-1)^k d_k(\beta^k). \qquad (1)$$

Let $\gamma_k = A_{k,*}(\beta^k) - (-1)^k \beta^k = A_{k,*}(\beta^k) + (-1)^{k+1}\beta^k$.

Then $d_k(\gamma_k) = 0 \Rightarrow \gamma_k \in \ker d_k = \underset{\substack{\uparrow \\ k\text{-cycles}}}{Z_k}(W_*(S^n))$