Definition 4.20 A topological space X is called *normal* if the following holds: For any closed subsets E and F with $E \cap F = \emptyset$ there are open subsets U and V such that $E \subset U$, $F \subset V$ and $U \cap V = \emptyset$.

Theorem 4.21. Every CW complex is a normal topological space.

proof. Rotman: Theorem 8.26.

Theorem 4.22. Let X be a CW complex and let Y be a CW subcomplex of X. Then X/Y is a CW complex.

proof. We first check that X/Y is Hausdorff: Let $p: X \to X/Y$ be the quotient map. Denote $p(Y)$ by $*$. Let $p(x), p(z) \in X/Y$, $p(x) \neq p(z)$. Assume first that $p(x) \neq *$, $p(z) \neq *$. Now X Hausdorff $\Rightarrow$ $X-Y$ Hausdorff $\Rightarrow$ $(X/Y) - \{*\} = p(X-Y)$ is Hausdorff ($p| : X-Y \to (X/Y)-\{*\}$ is a homeomorphism). Thus there are open neighbourhoods U of $p(x)$ and V of $p(z)$ in $(X/Y)-\{*\}$ with $U \cap V = \emptyset$. Since $(X/Y)-\{*\}$ is open in X/Y (Y, being a CW subcomplex, is closed in X), it follows that U and V are open in X/Y.

Assume then that $p(z) = *$. Then $x \notin Y$. Since X is normal by Theorem 4.22, there is a continuous map $f: X \to [0,1]$ with $f(x) = 0$ and $f(y) = 1$ $\forall y \in Y$. The map f induces a continuous map $\bar{f}: X/Y \to [0,1]$ making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & {[0,1]} \\ X/Y \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

(f continuous, p quotient map $\Rightarrow$ $\bar{f}$ continuous)

commute. Then $\bar{f}(p(x)) = 0$ and $\bar{f}(*) = 1$, $p(x) \in \bar{f}^{-1}[0,\tfrac{1}{2}) \subset X/Y$ and $* \in \bar{f}^{-1}(\tfrac{1}{2},1] \subset X/Y$, $\bar{f}^{-1}[0,\tfrac{1}{2}) \cap \bar{f}^{-1}(\tfrac{1}{2},1] = \emptyset$. $\therefore$ X/Y is Hausdorff.

Let:

(E, ϕ) = a CW decomposition of X
(E', ϕ') = — " — of Y, where $E' \subset E$, $\phi' \subset \phi$.

For every $n \geq 0$, let

E_n = the family of all n-cells in E
E'_n = — " — in E'.

Define the cells in X/Y as follows:

$$(X/Y)^0 = \{p(e) \mid e \in E_0 - E_0'\} \cup \{*\}$$

$$(X/Y)^n = \{p(e) \mid e \in E_n - E_n'\}, \quad n > 0.$$

Define the characteristic map of $p(e)$, $e \in E_n - E_n'$:

$$p \circ \phi_e : (D^k, S^{k-1}) \xrightarrow{\phi_e} (e \cup X^{(k-1)}, X^{(k-1)}) \xrightarrow{p} (p(e) \cup (X/Y)^{(k-1)}, (X/Y)^{(k-1)})$$

It remains to check that the four conditions in the definition of a CW complex are satisfied:

1) X is a disjoint union of its cells $\Rightarrow$ also X/Y is a disjoint union of its cells.

2) The map $p : (X, Y) \to (X/Y, *)$ is a relative homeomorphism. $\Rightarrow$ $p \circ \phi_e$ is a relat. homeom.

3) Let $e \in E_n - E_n'$. Then $\overline{p(e)} = p(\bar{e})$. Now, $\bar{e}$ is contained in a finite union of cells in X. $\Rightarrow$ $\overline{p(e)}$ is contained in a finite union of cells in X/Y.

4) Let $B \subset X/Y$. Assume $B \cap \overline{p(e)} \Subset \overline{p(e)}$ $\forall e \in E_n - E'_n$.
Then $p^{-1}(B \cap \overline{p(e)}) = p^{-1}(B) \cap p^{-1}(\overline{p(e)}) \Subset p^{-1}(\overline{p(e)})$
$= Y \cup \bar{e}$ $\forall e$. $\quad a = \bar{e}$

Let a be a cell in X. Then $p^{-1}(B) \cap \bar{a} = p^{-1}(B) \cap (Y \cup \bar{a}) \cap \bar{a}$.
Now, $p^{-1}(B) \cap (Y \cup \bar{a}) \Subset Y \cup \bar{a} \Rightarrow p^{-1}(B) \cap (Y \cup \bar{a}) \cap \bar{a} \Subset \bar{a}$. $\Rightarrow p^{-1}(B) \cap \bar{a} \Subset \bar{a}$
X has the weak topology determined by the $\bar{a}$
$\Rightarrow p^{-1}(B) \Subset X$. p quotient map $\Rightarrow B \Subset X/Y$ $\square$

<u>Lemma 4.23</u> Let (X, E) be a CW complex. Let $E' \subset E$, and let (Y, E') be a CW subcomplex of (X, E). Let $M \subset X^{(k)} - (X^{(k-1)} \cup Y)$. Assume M contains exactly one point of each k-cell in $X - Y$. Then $X^{(k-1)} \cup Y$ is a strong deformation retract of $(X^{(k)} \cup Y) - M$, for all $k \geq 1$.

<u>proof.</u> For every k-cell e, let $\{m_e\} = M \cap e$. Let $\phi_e : D^k \to X^{(k)} \cup Y$ be the characteristic map of e. We may assume that $\phi_e(0) = m_e$. Let

$$F : [(X^{(k)} \cup Y) - M] \times I \to (X^{(k)} \cup Y) - M,$$

$$(x, t) \mapsto \begin{cases} x & , \text{ if } x \in X^{(k-1)} \cup Y \\ \phi_e[(1-t)v + tv/\|v\|] & , \text{ if } x = \phi_e(v),\ v \neq 0 \\ & \quad \text{and } e \in E - E', \text{ e } k\text{-cell} \end{cases}$$

Then: 1) $F(x, 0) = x$ if $x \in X^{(k-1)} \cup Y$
$F(x, 0) = \phi_e(v)$ if $x = \phi_e(v), \ldots$
Then $F(x, 0) = x$ $\forall x$.

2) $F(x, 1) = x$, if $x \in X^{(k-1)} \cup Y$
$F(x, 1) = \phi_e(\underbrace{v/\|v\|}_{\in S^{k-1}}) \in X^{(k-1)}$, if $x = \phi_e(v), \ldots$
Thus $F(x, 1) \in X^{(k-1)} \cup Y$ $\forall x$.

3) $F(x, t) = x$ $\forall x \in X^{(k-1)} \cup Y$, $\forall t \in I$.

This part should be done more carefully.

It remains to check that F is continuous:

$X^{(k)} \cup Y$: weak topology determined by its cells

$(X^{(k)} \cup Y) - M$: weak topology determined by the cells in $X^{(k-1)} \cup Y$ and the punctured k-cells $e - \{m_e\}$.

$[(X^{(k)} \cup Y) - M] \times I$: product topology (I has the standard topology from $\mathbb{R}$).

I has a CW structure : 2 0-cells, 0 and 1
1 1-cell, $]0,1[$

It can be shown that the product topology on $[(X^{(k)} \cup Y) - M] \times I$ equals the weak topology determined by $e^* \times 0$, $e^* \times 1$ and $e^* \times]0,1[$, where e^* is a cell in $X^{(k-1)} \cup Y$ or a punctured cell $e - \{m_e\}$. (In fact: determined by the closures of these)

Then: The restriction of F to any of $e^* \times 0$, $e^* \times 1$ and $e^* \times]0,1[$ is continuous.

$\Rightarrow$ F is continuous. $\square$

<u>Theorem 4.24</u> Let X be a CW complex and let Y be a CW subcomplex of X. Then Y has an open neighborhood U in X such that Y is a strong deformation retract of U.

<u>Proof:</u> Let M_k consist of exactly one point from each k-cell in $X - Y$. Lemma 4.23 $\Rightarrow$ $X^{(k-1)} \cup Y$ is a strong deformation retract of $(X^{(k)} \cup Y) - M_k$ $\forall k \geq 1$. Hence, let

$$r_k : (X^{(k)} \cup Y) - M_k \to X^{(k-1)} \cup Y$$

be a strong deformation retraction. Define

$$U_0 = Y, \qquad U_k = r_k^{-1}(U_{k-1}), \quad k \geq 1.$$

Then, r_1 is a strong deformation retraction $\Rightarrow$

$$Y = U_0 \subset r_1^{-1}(U_0) = U_1.$$

Also: $X^{(0)}$ discrete & closed $\Rightarrow U_0 = Y \textcircled{c} X^{(0)} \cup Y \Rightarrow U_1 = r_1^{-1}(U_0) \textcircled{c} \underbrace{(X^{(1)} \cup Y) - M_1}_{\textcircled{c}\, X^{(1)} \cup Y}$

Inductively, $U_k = r_k^{-1}(U_{k-1}) \textcircled{c} (X^{(k)} \cup Y) - M_k \quad \forall k \geq 1,$

and $U_k \subset r_{k+1}^{-1}(U_k) = U_{k+1}$.

Then $Y \subset U = \bigcup_{k \geq 0} U_k$.

Assume $x \in U_k \cap X^{(k-1)}$. Then $r_k(x) \in U_{k-1} \subset X^{(k-1)} \cup Y$.

But $x \in X^{(k-1)} \Rightarrow r_k(x) = x$. Thus $x \in U_{k-1}$. $\Rightarrow U_k \cap X^{(k-1)} = U_{k-1} \cap X^{(k-1)}$.

Then $U \cap X^{(k)} = U_k \cap X^{(k)} \quad \forall k$.

Then $U \textcircled{c} X$, since $U \cap X^{(k)} = U_k \cap X^{(k)} \textcircled{c} X^{(k)} \quad \forall k$.

[Here you need the following:

<u>Exercise</u> Let X be a CW complex. Then U is open in X if and only if $X^{(n)} \cap U \textcircled{c} X^{(n)} \quad \forall n \geq 0$.]

Composition of strong deformation retractions is a strong deformation retraction. The sequence

$$\cdots \to U_{k+1} \xrightarrow{r_{k+1}} U_k \xrightarrow{r_k} U_{k-1} \to \cdots \to U_1 \xrightarrow{r_1} U_0 = Y$$

yields a strong deformation retraction $U_k \to Y \quad \forall k$.

Therefore, there are continuous maps $G_k : U_k \times I \to U_k$, for all $k \geq 1$ satisfying

$$G_k(x,0) = x \quad \forall\, x \in U_k$$
$$G_k(x,1) = r_1 \circ r_2 \circ \dots \circ r_k(x) \in Y \quad \forall\, x \in U_k$$
$$G_k(y,t) = y \quad \forall\, (y,t) \in Y \times I.$$

Moreover, $G_{k+1} | U_k \times I = G_k$. Let

$$H : U \times I \to U, \quad H(x,t) = G_k(x,t) \text{ if } (x,t) \in U_k \times I.$$

The topology on $U \times I$: the product topology. This equals the weak topology determined by $\{(U \times I) \cap (X^{(k)} \times I) \mid k \geq 1\} = \{\underset{U_k \cap X^{(k)}}{\cancel{G_k}} \times I \mid k \geq 1\}$.

Since the restrictions $H | U_k \times I$ are continuous $\forall k$, it follows that H is continuous. Thus Y is a strong deformation retract of U. $\square$

5. Cellular Homology

Definition 5.1. Let X be a topological space. A *filtration* of X is a sequence of subspaces X^n, $n \in \mathbb{Z}$, where $X^n \subset X^{n+1}$ $\forall n$. A filtration is called *cellular*, if the following hold:

1) $H_p(X^n, X^{n-1}) = 0 \quad \forall\, p \neq n$.

2) $\forall\, m \geq 0$ and $\forall$ continuous $\sigma : \Delta^m \to X$, $\exists\, n \in \mathbb{Z}$: $\operatorname{im}(\sigma) = \sigma(\Delta^m) \subset X^n$.

Notice: Let $m = 0$ and let $x \in X$. Then $\sigma : \underbrace{\Delta^0}_{\text{point } 1} \to X$, $1 \mapsto x$, is continuous. Thus $2 \Rightarrow x \in X^n$ for some n. $\Rightarrow X = \bigcup_n X^n$.

<u>Definition 5.2.</u> A topological space with cellular filtration is called a <u>cellular space.</u>

Let X and Y be cellular spaces. A continuous <u>map</u> $f: X \to Y$ is called <u>cellular</u>, if $f(X^n) \subset Y^n$, for all $n \in \mathbb{Z}$.

Cellular spaces and cellular maps form a category.

Later we show, that the filtration of a CW complex X by the skeletons $X^{(k)}$ is cellular. (Define $X^{(k)} = \emptyset$ for $k < 0$.)

Let X, Y be CW complexes and let $f: X \to Y$ be continuous. If

$$f(X^{(k)}) \subset Y^{(k)} \quad \forall \quad k \geq 0,$$

then f is called <u>cellular.</u>

<u>Definition 5.3.</u> Let X be a cellular space and let $k \geq 0$. Let

$$W_k(X) = H_k(X^k, X^{k-1}),$$

where H_k denotes singular homology. Define

$$d_k : W_k(X) \to W_{k-1}(X)$$

as $d_k = i_* \Delta$, where

$$\begin{tikzcd}
W_k(X) = H_k(X^k, X^{k-1}) \arrow[d, "\Delta"'] \arrow[dr, "d_k"] & \\
H_{k-1}(X^{k-1}) \arrow[r, "i_*"'] & H_{k-1}(X^{k-1}, X^{k-2}) = W_{k-1}(X)
\end{tikzcd}$$

Here Δ is the connecting homomorphism of the sequence

$$\cdots H_n(X^{k-1}) \to H_n(X^k) \to H_n(X^k, X^{k-1}) \xrightarrow{\Delta} H_{n-1}(X^{k-1}) \to \cdots,$$

for $n = k$, and i_* is induced by

$$i : (X^{k-1}, \phi) \hookrightarrow (X^{k-1}, X^{k-2}).$$

__Lemma 5.4.__ Let X be a cellular space. Then $(W_*(X), d)$ is a chain complex.

__proof.__ It suffices to check that $d_k d_{k+1} = 0$.

$$\begin{tikzcd}
H_{k+1}(X^{k+1}, X^k) \arrow[d, "\Delta"'] \arrow[dr, "d_{k+1}"] & & \\
H_k(X^k) \arrow[r, "i_*"'] & H_k(X^k, X^{k-1}) \arrow[d, "\Delta"'] \arrow[dr, "d_k"] & \\
& H_{k-1}(X^{k-1}) \arrow[r, "i_*"'] & H_{k-1}(X^{k-1}, X^{k-2})
\end{tikzcd}$$

Thus $d_k d_{k+1} = i_* \Delta i_* \Delta$. Now, there is the long exact sequence induced by $X^{k-1} \hookrightarrow X^k$:

$$\cdots \to H_n(X^{k-1}) \to H_n(X^k) \to H_n(X^k, X^{k-1}) \to H_{n-1}(X^{k-1}) \to \cdots$$

For $n = k$: $\cdots \to H_k(X^{k-1}) \to H_k(X^k) \xrightarrow{i_*} H_k(X^k, X^{k-1}) \to H_{k-1}(X^{k-1}) \to \cdots$

$\uparrow$

Δ: the connecting homomorphism

Exact sequence $\Rightarrow \Delta \circ i_* = 0$.
These Δ and i_* are exactly the maps in $d_k d_{k+1}$.
Then

$$d_k d_{k+1} = i_* \underbrace{\Delta i_*}_{=0} \Delta = 0. \quad \square$$

Theorem (Exact Sequence of the Triple (X, A, A')).

Let X be a topological space. Let $A \subset X$ and let $A' \subset A$. Then there is an exact sequence

$$\cdots \to H_n(A, A') \to H_n(X, A') \to H_n(X, A) \to H_{n-1}(A, A') \to \cdots$$

Moreover, if there is a commutative diagram of pairs of spaces

$$\begin{tikzcd}
(A, A') \arrow[r] \arrow[d] & (X, A') \arrow[r] \arrow[d] & (X, A) \arrow[d] \\
(B, B') \arrow[r] & (Y, B') \arrow[r] & (Y, B)
\end{tikzcd} \qquad (*)$$

then there is a commutative diagram of exact rows:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A, A') \arrow[r] \arrow[d] & H_n(X, A') \arrow[r] \arrow[d] & H_n(X, A) \arrow[r] \arrow[d] & H_{n-1}(A, A') \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_n(B, B') \arrow[r] & H_n(Y, B') \arrow[r] & H_n(Y, B) \arrow[r] & H_{n-1}(B, B') \arrow[r] & \cdots
\end{tikzcd} \qquad (**)$$

proof. Follows from the fact that there are short exact sequences of chain complexes:

$$0 \to S_*(A)/S_*(A') \to S_*(X)/S_*(A') \to S_*(X)/S_*(A) \to 0$$

and

$$0 \to S_*(B)/S_*(B') \to S_*(Y)/S_*(B') \to S_*(Y)/S_*(B) \to 0.$$

These short exact sequences induce long exact sequences in homologies. The diagram $(**)$ commutes by the naturality of the connecting homomorphism. $\square$

Let (X, A, B) be a triple of topological spaces. Then there is a commutative diagram

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\Delta"] \arrow[d, "\Delta'"'] & H_{n-1}(A) = H_{n-1}(A,\phi) \arrow[dl, "i_*"] \\
H_{n-1}(A,B) &
\end{tikzcd}$$

where $i : (A,\phi) \to (A,B)$ is the inclusion, Δ is the connecting homomorphism of the pair (X,A) and Δ' is the connecting homomorphism of the triple (X,A,B). We see this as follows:

Consider the following commutative diagram:

$$\begin{tikzcd}
(A,\phi) \arrow[r] \arrow[d, "i"'] & (X,\phi) \arrow[r] \arrow[d] & (X,A) \arrow[d, "id"] \\
(A,B) \arrow[r] & (X,B) \arrow[r] & (X,A)
\end{tikzcd}$$

The horizontal maps and the middle vertical map are inclusions of pairs. By the previous theorem, there is the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A,\phi) \arrow[r] \arrow[d] & H_n(X,\phi) \arrow[r] \arrow[d] & H_n(X,A) \arrow[r, "\Delta"] \arrow[d, "id"] & H_{n-1}(A,\phi) \arrow[r] \arrow[d, "i_*"] & \cdots \\
\cdots \arrow[r] & H_n(A,B) \arrow[r] & H_n(X,B) \arrow[r] & H_n(X,A) \arrow[r, "\Delta'"] & H_{n-1}(A,B) \arrow[r] & \cdots
\end{tikzcd}$$

This part gives the triangle above. $\square$

Notice: In the proof of Lemma 5.4 we did not use the assumption that the filtration is cellular. The claim is true for any filtration.

Lemma 5.5. Let X be a cellular space. Let $p \geq q$. Then

1) $H_n(X^p, X^q) = 0$ if $q \geq n$ or if $n > p$.

2) $H_n(X, X^q) = 0 \quad \forall\, q \geq n$.

3) $H_n(X, X^q) \cong H_n(X^{n+1}, X^q)$ if $q < n$.

4) $H_n(X, X^{-1}) \cong H_n(X, X^{-2}) \cong H_n(X, X^{-3}) \cong \quad \forall\, n$.

proof.

1) By induction on $p - q \geq 0$.

$p = q$: The identity map $Y \to Y$ induces the long exact sequence

$$\cdots \to H_n(Y) \xrightarrow{\cong} H_n(Y) \xrightarrow{\beta} H_n(Y, Y) \xrightarrow{\alpha} H_{n-1}(Y) \xrightarrow{\cong} H_{n-1}(Y) \to \cdots$$

Then $\operatorname{im}\alpha = 0 \Rightarrow \ker\alpha = H_n(Y, Y)$.
and $\ker\beta = H_n(Y) \Rightarrow \operatorname{im}\beta = 0$
$\Big\}\ \operatorname{im}\beta = \ker\alpha \Rightarrow H_n(Y, Y) = 0$

Thus $H_n(Y, Y) = 0$ for any topological space. In particular,

$H_n(X^p, X^q) \overset{p=q}{=} 0$ for $q \geq n$ or $n > p$.

Assume then $p - q > 0$. Consider the exact sequence of the triple (X^p, X^{q+1}, X^q):

$$\to \cdots \to H_n(X^{q+1}, X^q) \to H_n(X^p, X^q) \to H_n(X^p, X^{q+1}) \to \cdots$$

Assume first $n > p \geq q+1$: Definition of cellular filtration $\Rightarrow H_n(X^{q+1}, X^q) = 0$, since $n \neq q+1$.

$p-(q+1) < p-q \Rightarrow$ the induction assumption may be applied to $H_n(X^p, X^{q+1})$. Since $n > p$, it follows that $H_n(X^p, X^{q+1}) = 0$.

But then, also $H_n(X^p, X^q) = 0$. (By the exact seq. on p. 47)

Assume then $n \leq q$. Then $n \neq q+1$. Definition of cellular filtration $\Rightarrow H_n(X^{q+1}, X^q) = 0$.

$n \leq q < p \Rightarrow H_n(X^p, X^{q+1}) = 0$ by the induction assumption, since $n \leq q$.

Again, $H_n(X^p, X^q) = 0$.

2) Let $[z] \in H_n(X, X^q)$. Definition of cellular filtration $\Rightarrow \exists\, p \geq q : z \in S_n(X^p)$.

$$\Rightarrow [z] \in \operatorname{im}\big(\underbrace{H_n(X^p, X^q)}_{=0 \text{ if } q \geq n \text{ BY 1}} \to H_n(X, X^q)\big) = 0 \text{ if } q \geq n.$$

Then $H_n(X, X^q) = 0$ for $q \geq n$.

3) Consider the long exact sequence of the triple (X, X^{n+1}, X^q):

$$\to \cdots \to \underset{\substack{\| \\ 0 \text{ BY 2}}}{H_{n+1}(X, X^{n+1})} \to H_n(X^{n+1}, X^q) \to H_n(X, X^q) \to \underset{\substack{\| \\ 0 \text{ BY 2}}}{H_n(X, X^{n+1})} \to \cdots$$

THUS $H_n(X^{n+1}, X^q) \cong H_n(X, X^q)$, for $q \leq n+1$

(Here: $q = n, n+1$: $1 \Rightarrow H_n(X^{n+1}, X^q) = 0$
$2 \Rightarrow H_n(X, X^q) = 0$)

4) Let $q \leq -2$. Consider the exact sequence of the triple (X^{n+1}, X^{-1}, X^q):

$$\cdots \to H_n(X^{-1}, X^q) \to H_n(X^{n+1}, X^q) \to H_n(X^{n+1}, X^{-1}) \to H_{n-1}(X^{-1}, X^q) \to \cdots$$

Here:

$n \geq 0 \overset{i}{\Rightarrow} H_n(X^{-1}, X^q) = 0$

$n < 0 \Rightarrow H_n(X^{-1}, X^q) = 0$ (singular homology $= 0$ for $n < 0$)

Thus

$$H_n(X, X^q) \overset{3}{\cong} H_n(X^{n+1}, X^q) \cong H_n(X^{n+1}, X^{-1}) \overset{3}{\cong} H_n(X, X^{-1}). \quad \square$$

Theorem 5.6. Let X be a cellular space and let $k \geq 0$. Then

$$H_k(W_*(X)) \cong H_k(X, X^{-1}).$$

proof. Assume $k < n-1$. The triple (X^{n+1}, X^n, X^k) gives the following commutative diagram:

$$\begin{tikzcd} H_{n+1}(X^{n+1}, X^n) \arrow[r, "\Delta"] \arrow[d, "\Delta'"'] & H_n(X^n) \arrow[dl, "\lambda_*"] \\ H_n(X^n, X^k) & \end{tikzcd} \qquad (1)$$

Here: Δ = the connecting homomorphism of the pair (X^{n+1}, X^n)
Δ' = the connecting homomorphism of the triple (X^{n+1}, X^n, X^k)
λ_* : induced by the inclusion $\lambda: (X^n, \emptyset) \to (X^n, X^k)$

The diagram (where all maps are inclusions)

$$\begin{tikzcd} & (X^n, \emptyset) \arrow[dl, "\lambda"'] \arrow[d, "\mu"] \\ (X^n, X^k) \arrow[r, "j"'] & (X^n, X^{n-1}) \end{tikzcd} \qquad (2)$$

induces the commutative diagram

$$\begin{tikzcd} & H_n(X^n) \arrow[dl, "\lambda_*"'] \arrow[d, "\mu_*"] \\ H_n(X^n, X^k) \arrow[r, "j_*"'] & H_n(X^n, X^{n-1}) \end{tikzcd}$$

Combining diagrams 1 and 2 gives the commutative diagram

$$\begin{tikzcd} H_{n+1}(X^{n+1}, X^n) \arrow[r, "\Delta"] \arrow[d, "\Delta'"'] \arrow[dr, dashed, "d_{n+1}"] & H_n(X^n) \arrow[dl, "\lambda_*"'] \arrow[d, "\mu_*"] \\ H_n(X^n, X^k) \arrow[r, "j_*"'] & H_n(X^n, X^{n-1}) \end{tikzcd}$$

Above, $d_{n+1} = \mu_* \Delta$, by definition.

We end up getting the following commutative diagram, where the row and the columns are parts of exact sequence of triples.

$$\begin{tikzcd}
 & H_{n+1}(X^{n+1}, X^n) \arrow[d, "\Delta'"'] \arrow[dr, "d_{n+1}"] & & 0 \arrow[d] \\
0 \arrow[r] & H_n(X^n, X^k) \arrow[r, "j_*"] \arrow[d, "\alpha"'] & H_n(X^n, X^{n-1}) \arrow[r, "\Delta"] \arrow[dr, "d_n"'] & H_{n-1}(X^{n-1}, X^k) \arrow[d, "i_*"] \\
 & H_n(X^{n+1}, X^k) \arrow[d] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & 0 & &
\end{tikzcd}$$

($H_n(X^{n+1}, X^n) = 0$ by Lemma 5.5,
$H_{n-1}(X^{n-2}, X^k) = 0$ by Lemma 5.5)

<u>Exercise:</u> Show that the lower triangle in the diagram above commutes.

Then
$$\begin{aligned}
H_n(X, X^k) &\cong H_n(X^{n+1}, X^n) && \text{(Lemma 5.5 (3))} \\
&\cong H_n(X^n, X^k)/\ker\alpha \\
&= H_n(X^n, X^k)/\operatorname{im}\Delta' && \text{(The column on the left is exact.)} \\
&\cong \operatorname{im} j_*/\operatorname{im}(j_*\Delta') && (j_* \text{ injective}) \\
&= \ker\Delta/\operatorname{im}(j_*\Delta') && \text{(The row is exact.)} \\
&= \ker\Delta/\operatorname{im} d_{n+1} && (d_{n+1} = \mu_*\Delta = j_*\Delta') \\
&= \ker(i_*\Delta)/\operatorname{im} d_{n+1} && (i_* \text{ is an injection}) \\
&= \ker d_n/\operatorname{im} d_{n+1} && (d_n = i_*\Delta). \\
&= H_n(W_*(X)).
\end{aligned}$$

Then $H_n(X, X^k) \cong H_n(W_*(X))$, for all $k < n-1$.

Lemma 5.5 (4) $\Rightarrow$ $H_n(X, X^{-1}) \cong H_n(X, X^{-2}) \cong H_n(X, X^{-3}) \cong$ $\forall n$.

Thus $H_n(X, X^{-1}) \cong H_n(W_*(X))$, for all n. $\square$

Corollary 5.7. Let X be a cellular space with $X^{-1} = \phi$. Then
$$H_k(X) \cong H_k(W_*(X)), \text{ for all } k.$$
$\square$

Let then X be a CW complex.
Let

$X^{(k)}$ = the k-skeleton of X,
Y = a CW subcomplex of X,
$X^k_Y = X^{(k)} \cup Y$.

Then $X^{-1}_Y = Y$. If $Y = \phi$, then $X^{-1}_Y = \phi$.
Clearly,
$$X^k_Y \subset X^{k+1}_Y, \quad \text{for all } k.$$

Thus the X^k_Y define a filtration for X. We will soon show that this filtration is cellular.

Let $W_*(X, Y)$ denote the chain complex determined by the filtration X^k_Y. Then
$$W_k(X, Y) = H_k(X^k_Y, X^{k-1}_Y).$$

Let : X, X' be CW complexes
Y = a CW subcomplex of X
Y' = a CW subcomplex of X'

Let $f : X \to X'$ be a cellular map w.r.t. filtrations $X^{(k)}$ and $(X')^{(k)}$. Thus $f(X^{(k)}) \subset (X')^{(k)}$, for all $k \geq 0$.

Assume then that f is also a map of pairs

$$f:(X,Y)\to(X',Y').$$

Then f is cellular w.r.t. filtrations X^k_Y and $(X')^k_Y$. For every n, f induces a homomorphism

$$\bar{f}_n: H_n(X^k_Y, X^{k-1}_Y)\to H_n((X')^k_Y, (X')^{k-1}_Y),$$

in particular f induces

$$\bar{f}_k: W_k(X,Y)=H_k(X^k_Y, X^{k-1}_Y)\to H_k((X')^k_Y, (X')^{k-1}_Y)$$
$$= W_k(X',Y').$$

__Lemma 5.8.__ $f_*: W_*(X,Y)\to W_*(X',Y')$ is a chain map.

__proof.__ Exercise. □

Since f_* is a chain map, it induces homomorphisms

$$f_k: H_k(W_*(X,Y))\to H_k(W_*(X',Y')),$$

for all $k\geq 0$.

__Theorem 5.9.__ Let X be a CW complex and let Y be a CW subcomplex of X. Then:

1) The filtration of X by the X^k_Y is a cellular filtration.

2) Let $W_*(X,Y)$ be the corresponding cellular chain complex. Then, for all $k\geq 0$, there are isomorphisms

$$H_k(W_*(X,Y))\cong H_k(X,Y).$$