9) Let $\{(X_j, x_j) \mid j \in J\}$ be a family of pointed spaces. (Assume $\{x_j\} \in X_j$ $\forall j \in J$.) Let (Y, y_0) be a pointed space. Let $f_j : (X_j, x_j) \to (Y, y_0)$ be continuous pointed maps, for all $j \in J$. Define

$$f = \vee f_j : (\vee X_j, \ast) \to (Y, y_0),$$

as follows: If $x \in \vee X_j$, $x \neq \ast$, then $\exists!\, X_j : x \in X_j$. Define $f(x) = f_j(x_j)$. If $x = \ast$, define $f(\ast) = y_0$. Since $f|X_j = f_j$ for all $j \in J$ and since the topology on $\vee X_j$ equals the weak topology, it follows from Lemma 4.2, that f is continuous.

<u>Definition 4.3.</u> Let X be a topological space. Assume X is a disjoint union of cells, $X = \bigcup_{e \in E} e$ (where E is an indexing set). Let $k \geq 0$.

The <u>k-skeleton</u> of X is

$$X^{(k)} = \bigcup \{e \in E \mid \dim(e) \leq k\}.$$

Then $X^{(0)} \subset X^{(1)} \subset X^{(2)} \subset \dots$ and $X = \bigcup_{k \geq 0} X^{(k)}$.

<u>Definition 4.4.</u> A <u>CW complex</u> is an ordered triple (X, E, Φ), where X is a Hausdorff space, E is a family of cells in X, and $\Phi = \{\Phi_e \mid e \in E\}$ is a family of maps satisfying the following conditions:

1) X is the disjoint union $\bigcup\{e \mid e \in E\}$.

2) For every k-cell $e \in E$, the map

$$\Phi_e : (D^k, S^{k-1}) \to (e \cup X^{(k-1)}, X^{(k-1)})$$

is a relative homeomorphism.

3) If $e \in E$, then the closure $\bar{e}$ of e is contained in a finite union of cells in E.

4) X has the weak topology determined by $\{\bar{e} \mid e \in E\}$.

If (X, E, ϕ) is a CW-complex, then

1) X is a CW-space,

2) (E, ϕ) is a CW decomposition of X,

3) ϕ_e is the characteristic map of e.

Remark 4.5.

1) Condition 1 $\Rightarrow$ the cells $e \in E$ form a partition of X.

2) Condition 2 $\Rightarrow$ each k-cell $e \in E$ is obtained by attaching ~~a~~ k-cell to $X^{(k-1)}$ via the attaching map $\phi_e | S^{k-1}$.

3) Condition 3 is called closure finiteness.

4) C: closure finiteness
W: weak topology

5) Condition 4 implies the following:

a) A subset A of X is closed if and only if $A \cap \bar{e}$ is closed in $\bar{e}$ for every $\bar{e} \in E$.

b) Let Y be a topological space and let $f: X \to Y$ be a function. By Lemma 4.2, f is continuous if and only if $f|\bar{e} : \bar{e} \to Y$ is continuous for every $e \in E$.

6) A CW-space may have several different CW decompositions.

<u>Definition 4.6.</u> A CW complex (X, E, ϕ) is called <u>finite</u>, if E is a finite set.

<u>Example</u> A topological space is called a <u>topological manifold</u>, if it is locally euclidean and Hausdorff and has a countable basis (i.e., it is second-countable). It has been shown that every topological manifold has the same homotopy type as a CW complex. (see: A.T. Lundell and S. Weingram: Topology of CW complexes, van Nostrand-Reinhold, New York, 1968, p. 135.) Moreover, every compact topological manifold has the homotopy type of a finite CW complex (see: R.C. Kirby and L.C. Siebenmann: On the triangulation of manifolds and the Hauptvermutung, Bull. Amer. Math. Soc. 75, 1969, 742-749.

<u>Example</u> Consider S^n as a subspace of $\mathbb{R}^{n+1}$. For every $n \geq 1$, let

$$\phi : (D^n, S^{n-1}) \to (S^n, p), \quad x \mapsto (2\sqrt{1-\|x\|^2}\, x, 2\|x\|^2 - 1),$$

where $p = (0, \ldots, 0, 1) \in S^n$. Denote an i-cell by e^i. The map gives S^n a CW complex structure of 2 cells, $E = \{e^0, e^n\}$.

Notice, that by definition, $D^0 = \{0\}$ and $S^{-1} = \emptyset$. Thus also S^0 has a CW decomposition of 2 cells $\{e_1^0, e_2^0\}$.

<u>Example</u> The real line $\mathbb{R}$ admits the structure of 1-dimensional CW complex with the integers as 0-cells and the intervals $[n, n+1]$, $n \in \mathbb{Z}$ as 1-cells.

Example Every smooth manifold has a CW structure (in fact, every smooth manifold can be triangulated). Triangulation means that the smooth manifold has the structure of a simplicial complex. Any simplicial complex is a CW-complex whose n-cells are the n-simplices.

Example $\mathbb{R}P^n$ has a CW-structure with one cell of dimension k, $0 \le k \le n$. $\mathbb{C}P^n$ has a CW structure with one cell of dimension $2k$, $0 \le k \le n$.

Definition 4.7. Let (X, E, Φ) be a CW-complex, let $E' \subset E$. Define

$$|E'| = \bigcup\{e \mid e \in E'\} \subset X \quad \text{and} \quad \Phi' = \{\phi_e \mid e \in E'\}.$$

If $\operatorname{im}(\phi_e) \subset |E'|$ for every $e \in E'$, we call $(|E'|, E', \Phi')$ a CW subcomplex of (X, E, Φ).

Let $(|E'|, E', \Phi)$ be a CW subcomplex of a CW complex (X, E, Φ). Then $(|E'|, E', \Phi)$ is a CW complex, $|E'|^{(k)} = X^{(k)} \cap |E'|$, for all $k \ge 0$.

Lemma 4.8. Let (X, E, Φ) be a CW complex, and let $e \in E$ be a k-cell, $k > 0$, with characteristic map ϕ_e. Then $\bar{e} = \operatorname{im}\phi_e = \phi_e(D^k)$.

Proof. 1) $\phi_e(D^k) = \phi_e(\overline{D^k - S^{k-1}}) \underset{\substack{\uparrow \\ \phi_e \text{ continuous}}}{\subset} \overline{\underbrace{\phi_e(D^k - S^{k-1})}_{e}} = \bar{e}$.

2) D^k compact, ϕ_e continuous $\Rightarrow$ $\phi_e(D^k)$ compact.
X Hausdorff $\Rightarrow$ $\phi_e(D^k)$ is closed in X.
Also, $e = \phi_e(D^k - S^{k-1}) \subset \phi_e(D^k)$
$\Rightarrow$ $\bar{e} \subset \phi_e(D^k)$.

1 & 2 $\Rightarrow$ $\bar{e} = \phi_e(D^k)$. $\square$

Example Let (X, E, Φ) be a CW complex and let $E' \subset E$. By Lemma 4.8, $(|E'|, E', \Phi')$ is a CW subcomplex of (X, E, Φ) if and only if $\bar{e} \subset |E'|$ for every $e \in E'$. Fix k and assume E' is a family of k-cells in E. Then $|E'| \cup X^{(k-1)}$ is a CW subcomplex, since by property 2 of Definition 4.4, $\bar{e} \subset e \cup X^{(k-1)}$. Choosing $E' = \emptyset$, we see that the k-skeleton $X^{(k)}$ is a CW-subcomplex of X. (Notice: Instead of saying that (X, E, Φ) is a CW complex, we often say that X is a CW complex.)

Hausdorff

Lemma 4.9. Let (X, E, Φ) be an ordered triple satisfying the first 2 conditions of the definition of CW complex, i.e.,

1) X is the disjoint union $\bigcup\{e \mid e \in E\}$,

2) For every k-ball $e \in E$, the map

$$\Phi_e : (D^k, S^{k-1}) \to (e \cup X^{(k-1)}, X^{(k-1)})$$

is a relative homeomorphism.

Let

$$\varphi = \coprod_{e \in E} \Phi_e : \coprod_{e \in E} D^{n(e)} \to X.$$

Then X has the weak topology determined by the family $\{\bar{e} \mid e \in E\}$ if and only if φ is an identification.

proof. Assume first that X has the weak topology. Notice that the conclusion of Lemma 4.8 holds (only conditions 1 and 2 of the definition of CW complex were used in the proof). Thus, for every $e \in E$, the map $\Phi_e : D^{n(e)} \to \bar{e}$ is a surjection.

Now, φ is a continuous surjection. To show that φ is an identification, it suffices to show that if $C \subset X$ and $\varphi^{-1}(C)$ is closed in $\coprod_e D^{n(e)}$, then C is closed in X. Thus, assume $C \subset X$ and $\varphi^{-1}(C)$ is closed in $\coprod_e D^{n(e)}$. Then $\varphi^{-1}(C) \cap D^{n(e)}$ is closed in $D^{n(e)}$. Since $D^{n(e)}$ is compact, it follows that $\varphi^{-1}(C) \cap D^{n(e)}$ is compact. But

$$\begin{aligned} \varphi^{-1}(C) \cap D^{n(e)} &= \phi_e^{-1}(C) \cap D^{n(e)} \\ &= \phi_e^{-1}(C) \cap \phi_e^{-1}(\bar{e}) \quad (\text{Lemma } 4.8) \\ &= \phi_e^{-1}(C \cap \bar{e}). \end{aligned}$$

Since $\phi_e : D^{n(e)} \to \bar{e}$ is a surjection, it follows that $C \cap \bar{e} = \phi_e(\phi_e^{-1}(C \cap \bar{e}))$. Since $\phi_e^{-1}(C \cap \bar{e})$ is compact and ϕ_e is continuous, it follows that $C \cap \bar{e}$ is compact. Thus $C \cap \bar{e}$ is closed in $\bar{e}$. Since X has the weak topology determined by $\{\bar{e} \mid e \in E\}$, it follows that C is closed in X. Therefore, φ is an identification.

Assume then that φ is an identification. Let $C \subset X$, and assume $C \cap \bar{e}$ is closed in $\bar{e}$ for all $e \in E$. Then $\varphi^{-1}(C) \cap D^{n(e)}$ is closed in $D^{n(e)}$ for all $e \in E$. Since $\coprod_e D^{n(e)}$ has the weak topology determined by $\{D^{n(e)} \mid e \in E\}$, it follows that $\varphi^{-1}(C)$ is closed in $\coprod_e D^{n(e)}$. Since φ is an identification, C is closed in X. Thus X has the weak topology determined by $\{\bar{e} \mid e \in E\}$.

$\square$

<u>Lemma 4.10.</u> Let (X, E, ϕ) be a CW complex, and let E' be a finite subset of E. Then $|E'|$ is a CW subcomplex of X if and only if $|E'|$ is closed in X.

<u>proof.</u> Assume $|E'|$ is a finite subcomplex of X. By Lemma 4.8, $\bar{e} = \phi_e(D^{n(e)}) \subset |E'|$ for every $e \in E'$. Then

$$|E'| = \bigcup\{e \mid e \in E'\} = \bigcup\{\bar{e} \mid e \in E'\}$$

is closed as a finite union of closed sets.
Assume then that $|E'|$ is closed. Let $e \in E'$. Then $e \in |E'|$ and since $|E'|$ is closed, it follows that $\phi_e(D^{n(e)}) = \bar{e} \subset |E'|$. Thus $|E'|$ is a CW subcomplex of X. $\square$

Lemma 4.11. Let (X, E, ϕ) be a CW complex and let $e \in E$. Then $\bar{e}$ is contained in a finite CW subcomplex of X.

proof. We prove the claim by induction on $n = \dim(e)$.
Clearly, the claim is true if $n = 0$. Assume then $n > 0$. By Lemma 4.8,

$$\bar{e} - e = \phi_e(D^n) - e \subset (e \cup X^{(n-1)}) - e \subset X^{(n-1)}. \quad (*)$$

Condition 3 in Def. 4.4 $\Rightarrow$ $\bar{e}$ intersects only finitely many cells other than e. Let such cells be $e_1, \ldots, e_m$. By $(*)$, $\dim(e_i) \leq n-1$, for all $i = 1, \ldots, m$. By induction, for each $i = 1, \ldots, m$, there is a finite CW subcomplex X_i containing $\bar{e}_i$. By Lemma 4.10, each X_i is closed. Then $\bar{e} \subset e \cup X_1 \cup \cdots \cup X_m$, which is a union of finitely many cells and also a closed subset of X (each X_i is closed and $\bar{e} \subset e \cup X_1 \cup \cdots \cup X_m \Rightarrow e \cup X_1 \cup \cdots \cup X_m = \bar{e} \cup X_1 \cup \cdots \cup X_m$ is closed as a finite union of closed sets). By Lemma 4.10, $e \cup X_1 \cup \cdots \cup X_m$ is a CW subcomplex.
$\square$

Theorem 4.12. Let (X, E, ϕ) be a CW complex. Then:

1) Every compact subset of X is contained in a finite CW subcomplex of X.

2) A CW space X is compact, if and only if (X, E, ϕ) is a finite CW complex for every CW decomposition (E, ϕ) of X.

proof. We prove the first claim. The second claim follows from the first one. Let K be a compact subset of X. For every $e \in E$ s.t. $K \cap e \neq \emptyset$, choose $a_e \in K \cap e$. Let $A = \{a_e\}$. Lemma 4.11 $\Rightarrow$ $\exists$ a finite CW subcomplex X_e of X : $\bar{e} \subset X_e$. Then $A \cap \bar{e} \subset A \cap X_e$ is finite, and hence $A \cap \bar{e}$ is closed in $\bar{e}$. Since X has the weak topology determined by $\{\bar{e} \mid e \in E\}$, it follows that A is closed in X. Similarly, every subset of A is closed in X. Thus A is a discrete set. Since $A \subset K$, A is closed and K is compact, it follows that A is compact. Thus A must be a finite set. It follows that K can intersect only finitely many $e \in E$. Let such cells be $e_1, \dots, e_m$. Lemma 4.11 $\Rightarrow$ for every $i = 1, \dots, m$ there is a finite CW subcomplex X_i of X : $\bar{e}_i \subset X_i$. Thus K is contained in the finite subcomplex $\bigcup_{i=1}^{m} X_i$ of X. $\square$

Lemma 4.13. Let (X, E, Φ) be a CW complex and let $A \subset X$. Then the following are equivalent:

1) A is closed in X.

2) $A \cap X'$ is closed in X' for every finite CW subcomplex X' of X.

proof. If A is closed in X, then $A \cap X'$ is closed in X'. Thus, clearly 1 implies 2. Assume then that condition 2 holds. Let $e \in E$. By Lemma 4.11, there is a finite CW subcomplex X_e : $\bar{e} \subset X_e$. By assumption, $A \cap X_e$ is closed in X_e. Then $A \cap \bar{e} = (A \cap X_e) \cap \bar{e}$ is closed in X_e, and hence $A \cap \bar{e}$ is also closed in $\bar{e}$. Since X has the weak topology generated by $\{\bar{e} \mid e \in E\}$, it follows that A is closed in X. $\square$

Notice that by Lemma 4.13 a CW complex has the weak topology generated by its finite CW subcomplexes.

Theorem 4.14. Let (X, E, ϕ) be a CW complex, and let $E' \subset E$. Then $|E'|$ is a CW subcomplex of X if and only if $|E'|$ is closed.

proof. Assume first that $|E'|$ is closed. Let $e \in E'$. Then $\bar{e} \subset |E'|$. Thus $|E'|$ is a CW subcomplex (Lemma 4.8).

Assume then that $|E'| = X'$ is a CW subcomplex. Let Y be a finite CW subcomplex of X. Then $X' \cap Y$ is a union of finitely many cells, write $X' \cap Y = e_1 \cup \dots \cup e_m$. Since $X' \cap Y$ is a CW subcomplex, it follows that $\bar{e}_i \subset X' \cap Y \ \forall i \in \{1, \dots, m\}$. Thus $X' \cap Y = \bar{e}_1 \cup \dots \cup \bar{e}_m$. It follows that $X' \cap Y$ is closed in Y. (In fact, $X' \cap Y = \bar{e}_1 \cup \dots \cup \bar{e}_m$ is closed in X.) Lemma 4.13. $\Rightarrow$ $|E'| = X'$ is closed. □

Corollary 4.15. Let (X, E, ϕ) be a CW complex. Let $n > 0$. Let E' be a family of n-cells in E. Then

1) $X' = |E'| \cup X^{(n-1)}$ is closed in X.

2) The n-skeleton $X^{(n)}$ is closed in X, $n \geq 0$

3) Every n-cell e is open in $X^{(n)}$, $n \geq 0$.

4) $X^{(n)} - X^{(n-1)}$ is open in $X^{(n)}$.

proof. 1) Example on p. 32 $\Rightarrow$ X' is a CW subcomplex of X. Thus X' is closed.

2) $n>0$: Choose $E'=\emptyset$ in part 1. The claim follows from part 1. $n=0$: Exercise.

3) Let E' consist of every n-cell in E except e.
$1 \Rightarrow X' = |E'| \cup X^{(n-1)} \subseteq X$
$\Rightarrow e = X^{(n)} - X' \subseteq X^{(n)}$

4) Follows from part 3. □

<u>Theorem 4.16.</u> Let X be a CW complex. Then:

1) Every path component of X is a CW subcomplex. Thus every path component of X is closed.

2) Every path component of X is both open and closed!

3) The path components of X are the components of X!

4) X is connected if and only if X is path-connected.

<u>Proof.</u> 1) X is a disjoint union of cells, each cell is path connected. $\Rightarrow$ each path component is a union of cells. Let A be a path component. Let e be an n-cell and assume $e \subset A$. Then $\bar{e} = \Phi_e(D^n)$ is path connected, thus $\bar{e} \subset A$. Thus A is a CW subcomplex, in particular, A is closed.

2) Let A be a path component of X. Let B be the union of all path components of X other than A. Then, by 1, B is a union of CW subcomplexes.
$\Rightarrow$ B is a CW subcomplex $\Rightarrow$ B is closed
$\Rightarrow$ $A = X - B$ is open.

3) Let A be a path component of X. Then $A \subset Y$, where Y is some component of X. Now, A is both open and closed in X.
$\Rightarrow$ $Y = A \cup (Y - A)$ is a union of open subsets A and $Y - A$ of Y, where $A \cap (Y - A) = \phi$. Then Y is not connected unless $Y - A = \phi$, i.e., $A = Y$.

4) Follows from 3. $\square$

Recall the following definition:

<u>Definition 4.17.</u> A topological space X is called <u>locally path connected</u> if the following holds: For every $x \in X$ and for every open neighborhood U of x, there is an open subset V of X s.t. $x \in V \subset U$ and any two points in V can be joined by a path in U.

<u>Lemma 4.18.</u> A topological space X is locally path connected if and only if the following holds: For every $x \in X$ and for every open neighborhood U of x, there is an open path connected subset V of X s.t. $x \in V \subset U$.

<u>proof.</u> See the lecture notes "Introduction to Algebraic topology" fall 2015, or see Rotman: Corollary 1.19. $\square$

<u>Theorem 4.19.</u> Every CW complex is locally path connected.

<u>proof.</u> Rotman; Theorem 8.25. $\square$

Example (by Dowker, see Hatcher's appendix)

The following is an example of two CW complexes X and Y s.t. the product topology on $X \times Y$ is not the same as the CW topology on $X \times Y$:

Let $X = \bigvee_s I_s$, I_s = a copy of $[0,1]$
s: ranges over all infinite sequences $s = (s_1, s_2, \ldots)$ of positive integers $(s \in S)$
wedge sum formed at $0 \in [0,1]$

Let $Y = \bigvee_j I_j$, I_j = a copy of $[0,1]$
j: ranges over all positive integers $(j \in J)$

Let $p_s = (1/s_j, 1/s_j) \in I_s \times I_j$

(i.e., choose s and j, then s_j = the j^{th} element in s)

Let $P = \{p_s \mid s \in S, j \in J\}$.

$\forall s, j$: $p_s \in [0,1] \times [0,1] \leftarrow$ closed 2 cell of $X \times Y$

$P \cap$ (any closed ~~2 cell~~ of $X \times Y$) is a point or $\emptyset$
$\Rightarrow$ P is closed in the CW topology of $X \times Y$.

Let x_0 = the common end point of the intervals I_s,
y_0 = — " — I_j.

Let $\overline{P}$ denote the closure of P in the product topology of $X \times Y$. We show that $(x_0, y_0) \in \overline{P}$. Since $(x_0, y_0) \notin P$, it then follows that P is not closed in the product topology.

A basic open neighborhood of (x_0, y_0) in the product topology has the form $U \times V$, where

$$U = \bigvee_s [0, a_s), \quad V = \bigvee_j [0, b_j).$$

Enough to show: $(U \times V) \cap P \neq \emptyset$.

Let $t = (t_1, t_2, \dots)$ be a sequence of positive integers, where $t_j > j$ and $t_j > 1/\ell_j$ $\forall j$.

Choose $k \in \mathbb{N}$: $k > a_t^{-1} = 1/a_t$.

Then $t_k > k > 1/a_t$. $\Rightarrow a_t > 1/t_k$.
Also, $\ell_k > 1/t_k$.

Thus $(1/t_k, 1/t_k) \in P$ and $(1/t_k, 1/t_k) \in [0, a_t) \times [0, \ell_k)$
$\subset U \times V$.

$\square$

<u>Definition 4.20</u> A topological space X is called <u>normal</u> if the following holds: For any closed subsets E and F with $E \cap F = \emptyset$ there are open subsets U and V such that $E \subset U$, $F \subset V$ and $U \cap V = \emptyset$.

<u>Theorem 4.21.</u> Every CW complex is a normal topological space.

<u>proof.</u> Rotman: Theorem 8.26.

<u>Theorem 4.22.</u> Let X be a CW complex and let Y be a CW subcomplex of X. Then X/Y is a CW complex.

<u>proof.</u> We first check that X/Y is Hausdorff: Let $p: X \to X/Y$ be the quotient map. Denote $p(Y)$ by $*$. Let $p(x), p(z) \in X/Y$, $p(x) \neq p(z)$. Assume first that $p(x) \neq *$, $p(z) \neq *$. Now X Hausdorff $\Rightarrow X - Y$ Hausdorff $\Rightarrow (X/Y) - \{*\} = p(X-Y)$ is Hausdorff ($p| : X - Y \to (X/Y) - \{*\}$ is a homeomorphism). Thus there are open neighborhood U of $p(x)$ and V of $p(z)$ in $(X/Y) - \{*\}$ with $U \cap V = \emptyset$. Since $(X/Y) - \{*\}$ is open in X/Y (Y, being a CW subcomplex, is closed in X), it follows that U and V are open in X/Y.

Assume then that $p(z) = *$. Then $x \notin Y$. Since X is normal by Theorem 4.22, there is a continuous map $f: X \to [0,1]$ with $f(x) = 0$ and $f(y) = 1$ $\forall y \in Y$. The map f induces a continuous map $\bar{f}: X/Y \to [0,1]$ making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & {[0,1]} \\ X/Y \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

(f continuous, p quotient map $\Rightarrow \bar{f}$ continuous)

commute. Then $\bar{f}(p(x)) = 0$ and $\bar{f}(*) = 1$, $p(x) \in \bar{f}^{-1}[0, \frac{1}{2}) \overset{\circ}{\subset} X/Y$ and $* \in \bar{f}^{-1}(\frac{1}{2}, 1] \overset{\circ}{\subset} X/Y$, $\bar{f}^{-1}[0,\frac{1}{2}) \cap \bar{f}^{-1}(\frac{1}{2}, 1] = \emptyset$. $\therefore X/Y$ is Hausdorff.