Thus, we must show that for $u, v \in D^n \amalg Y$, $u \sim v$ if and only if $h(u) = h(v)$. Assume $u \sim v$. We may assume that $u \in S^{n-1}$. Then

1) $v \in Y$ and $v = \phi(u)$, or

2) $v \in S^{n-1}$ and $\phi(v) = \phi(u)$.

In both cases, $h(u) = h(v)$. Assume then that $h(u) = h(v)$. Then, either $u, v \in D^n$, $u, v \in Y$, or $u \in D^n$ and $v \in Y$ (similarly, $v \in D^n$ and $u \in Y$). If $u, v \in Y$, then $u = h(u) = h(v) = v$. Hence $u \sim v$. If $u \in D^n$ and $v \in Y$, then $\phi(u) = h(u) = h(v) = v$. Hence $u \sim v$. Assume $u, v \in D^n$. Since, by assumption, the restriction $\phi | D^n - S^{n-1} : D^n - S^{n-1} \to (e \cup Y) - Y = e$ is a homeomorphism, it follows that $u, v \in S^{n-1}$ and $\phi(u) = \phi(u)$ $= \phi(v) = \phi(v)$. Thus $u \sim v$. It now follows from Lemma 2.6, that the map

$$g : Y_\phi \to e \cup Y, \ [u] \mapsto (\phi \amalg id_Y)(u) = h(u),$$

is a homeomorphism. $\square$

The relative homeomorphism

$$\phi : (D^n, S^{n-1}) \to (e \cup Y, Y)$$

is also called a characteristic map.

3. Homology and Attaching Cells

<u>Definition 3.1.</u> Let X be a Hausdorff space. We call X <u>locally compact</u>, if for every $x \in X$ and for every open neighborhood U of x, there is an open subset W of X such that $x \in W \subset \overline{W} \subset U$ and $\overline{W}$ is compact.

Lemma 3.2. Let X, X' and Z be topological spaces. Assume that Z is locally compact Hausdorff. Let $p: X \to X'$ be an identification. Then also

$$p \times id_Z : X \times Z \to X' \times Z$$

is an identification.

proof. Since p and id_Z are continuous, also $p \times id_Z$ is continuous. Thus $(p \times id_Z)^{-1}(U')$ is open in $X \times Z$ for every open subset U' of $X' \times Z$.

Assume then that $U' \subset X' \times Z$ and that $U = (p \times id_Z)^{-1}(U')$ is open. It suffices to show that U' is open. Let $(x', z) \in U'$, and let $x \in X : p(x) = x'$. Then $(p \times id_Z)(x, z) = (x', z)$, which implies that $(x, z) \in U$. Since U is open in $X \times Z$, x has an open neighbourhood V in X and z has an open neighbourhood J in Z with $(x, z) \in V \times J \subset U$. Since Z is locally compact, there is an open subset W of Z such that $z \in W \subset \overline{W} \subset J$ and $\overline{W}$ is compact. Then $\{x\} \times \overline{W} \subset V \times J \subset U$. Let

$$A = \{ \alpha \in X \mid \{\alpha\} \times \overline{W} \subset U \}.$$

Then $x \in A$. We show that A is open in X. Let $\alpha \in A$. For every $\xi \in \overline{W}$, there are open subsets L_ξ of X and N_ξ of Z with $(\alpha, \xi) \in L_\xi \times N_\xi \subset U$. Since $\overline{W}$ is compact, the open cover $\{ N_\xi \mid \xi \in \overline{W} \}$ of $\overline{W}$ has a finite subcover $\{N_1, \dots, N_m\}$. Then, for every $i \in \{1, \dots, m\}$, $L_i \times N_i \subset U$, where we define $L_i = L_\xi$, if $N_i = N_\xi$. Also, $\alpha \in \bigcap_{i=1}^m L_i$ and $\overline{W} \subset \bigcup_{i=1}^m N_i$. Thus $\bigcap_{i=1}^m L_i = L$ is open and $\alpha \in L$, $L \times N_i \subset U$ for every i. Hence

$$L \times \overline{W} \subset \bigcup_{i=1}^m (L \times N_i) \subset U.$$

Consequently, $\alpha \in L \subset A$. It follows that A is open in X.

For $\beta \in X$, $\{\beta\} \times \overline{W} \subset U = (p \times id_Z)^{-1}(U')$ if and only if $\{p(\beta)\} \times \overline{W} \subset U'$. In particular, $\beta \in A$ if and only if $\{p(\beta)\} \times \overline{W} \subset U'$. Clearly, $A \subset p^{-1}(p(A))$. Assume $\beta \in p^{-1}(p(A))$. Then $p(\beta) \in p(A)$. Then $p(\beta) = p(\alpha)$, for some $\alpha \in A$. Hence $\{p(\beta)\} \times \overline{W} = \{p(\alpha)\} \times \overline{W} \subset U'$. But this now implies that $\beta \in A$. Therefore, $p^{-1}(p(A)) \subset A$ and, consequently, $A = p^{-1}(p(A))$. Since p is an identification and A is open, it follows that $p(A)$ is open. Thus $p(A) \times W \subset U'$ is an open neighborhood of (x', z). Hence, U' is open. $\square$

<u>Lemma 3.3.</u> (Tube Lemma) Let X and Y be topological spaces. Assume Y is compact. Let $x_0 \in X$ and let U be an open subset of $X \times Y$ s.t. $\{x_0\} \times Y \subset U$. Then x_0 has an open neighborhood L in X with $\{x_0\} \times Y \subset L \times Y \subset U$.

<u>proof.</u> Similar to the proof of Lemma 3.2. $\square$

Recall the following: Let $f, g : (X, A) \to (Y, B)$ be continuous maps of pairs. If there is a continuous map

$$F : (X \times I, A \times I) \to (Y, B)$$

with $F_0 = f$ and $F_1 = g$, we say that f and g are homotopic and F is a homotopy from f to g. (We may write $F : f \simeq g \mod A$. Notice that $f|A$ does not need to equal $g|A$ and $F(a,t)$, $a \in A$, may depend on t.)

<u>Corollary 3.4.</u> Let $f, g : (X, A) \to (Y, B)$ be homotopic. Then the induced maps $\bar{f}, \bar{g} : (X/A, *) \to (Y/B, *)$ are homotopic (as maps of pointed spaces, here $p : X \to X/A$ and $* = p(A)$, similarly for B...).

proof. Let $F: (X \times I, A \times I) \to (Y, B)$ be a homotopy from f to g. Then F induces a function $\bar{F}: (X/A) \times I \to Y/B$ and the diagram

$$\begin{tikzcd} X \times I \arrow[r, "F"] \arrow[d, "p \times id"'] & Y \arrow[d, "q"] \\ (X/A) \times I \arrow[r, "\bar{F}"'] & Y/B \end{tikzcd}$$

commutes, where $p: X \to X/A$ and $q: Y \to Y/B$ are quotient maps and $id: I \to I$ is the identity. Since F and q are continuous, it follows that $\bar{F} \circ (p \times id) = q \circ F$ is continuous. By Lemma 3.2, $p \times id$ is an identification. It follows that $\bar{F}$ is continuous. $\square$

The following theorem relates the singular homologies of Y and Y_f:

Theorem 3.5. Let Y be a Hausdorff space, and let Y_f be obtained from Y by attaching an n-cell via f, where $n \geq 1$. Then there exists an exact sequence

$$\cdots \to H_p(S^{n-1}) \xrightarrow{f_*} H_p(Y) \xrightarrow{i_*} H_p(Y_f) \to H_{p-1}(S^{n-1}) \to \cdots$$

$$\cdots \to H_0(S^{n-1}) \to \mathbb{Z} \oplus H_0(Y) \to H_0(Y_f) \to 0,$$

where $i: Y \hookrightarrow Y_f$ is the inclusion.

proof. Let $p: D^n \amalg Y \to Y_f$ be the quotient map. Let

$$\Phi = p | D^n : (D^n, S^{n-1}) \to (Y_f, Y)$$

be the characteristic map. Let $e = \Phi(D^n - S^{n-1})$ be an open n-cell in Y_f, and let U' be the open n-disk in D^n, with radius $\frac{1}{2}$ and origin as the center. Now ϕ is a relative homeomorphism and e is open in $Y_\phi \Rightarrow U = \phi(U')$ is open in Y_ϕ. Let $V = Y_\phi - \phi(0)$. Then $\{U, V\}$ is an open cover of Y_ϕ. It follows from Excision II (see the notes from Fall 2015) that there is an exact Mayer-Vietoris sequence

$$\cdots \to H_p(U \cap V) \to H_p(U) \oplus H_p(V) \to H_p(Y_\phi) \to H_{p-1}(U \cap V) \to \cdots \quad (*)$$

Now, U is contractible $\Rightarrow H_p(U) = 0$ for $p > 0$. The homomorphisms

$$H_p(U \cap V) \to H_p(V) \quad \text{and} \quad H_p(V) \to H_p(Y_\phi), \quad p > 0,$$

are induced by inclusions.

$U \cap V$: homeomorphic to an open punctured disk of dimension n $\Rightarrow$ $U \cap V$ has the same homotopy type as S^{n-1}

V: Let

$$F : V \times I \to V, \ (v, t) \mapsto \begin{cases} v & , \text{ if } v \in Y \\ \phi\left((1-t)z + t\frac{z}{\|z\|}\right) & , \text{ if } v = \phi(z) \in e. \end{cases}$$

Since $Y \cap e = \emptyset$, it follows that F is well-defined. Then:

1) $F(v, t) = v$, for all $(v, t) \in Y \times I$.

2) Let $t = 1$. If $v \in Y$, then $F(v, 1) = v \in Y$. If $v = \phi(z) \in e$, then $F(v, 1) = \phi(\underbrace{\tfrac{z}{\|z\|}}_{\in S^{n-1}}) \in Y$. Thus $F(V \times \{1\}) \subset Y$.

3) Let $t = 0$. If $v \in Y$, then $F(v, 0) = v \in Y$. If $v = \phi(z) \in e$, then $F(v, 0) = \phi(z + 0) = \phi(z) = v$. Thus $F_0 = id_V$.

We show that F is continuous. Properties 1, 2, 3 will then imply that Y is a strong deformation retract of V.

Let
$$p' : (D^n - \{0\}) \amalg Y \to V$$
be the restriction of $p : D^n \amalg Y \to Y_\phi$. Let
$$h : \big((D^n - \{0\}) \amalg Y\big) \times I \to (D^n - \{0\}) \amalg Y,$$
$$\begin{cases} (x,t) \mapsto (1-t)x + t\frac{x}{\|x\|}, & \text{for } (x,t) \in (D^n - \{0\}) \times I \\ (y,t) \mapsto y, & \text{for } (y,t) \in Y \times I. \end{cases}$$

Consider the following commutative diagram:

$$\begin{tikzcd}
\big((D^n - \{0\}) \amalg Y\big) \times I \arrow[r, "h"] \arrow[d, "p' \times id"'] \arrow[dr, "p' \circ h"] & (D^n - \{0\}) \amalg Y \arrow[d, "p'"] \\
V \times I \arrow[r, "F"'] & V = Y_\phi - \{\phi(0)\}
\end{tikzcd}$$

By Lemma 3.2, $p' \times id$ is an identification. Since $p' \circ h$ is constant on the fibers of $p' \times id$, it follows that F is continuous. Thus Y is a strong deformation retract of V, and Y and V have the same homotopy type.

The Mayer-Vietoris sequence $(*)$ now becomes

$$\cdots \to H_p(S^{n-1}) \to H_p(Y) \to H_p(Y_\phi) \to H_{p-1}(S^{n-1}) \to \cdots$$

$$\cdots \to H_0(S^{n-1}) \to \underset{\substack{\cong \\ H_0(U),\ \text{since } U \text{ is} \\ \text{pathconnected}}}{\mathbb{Z}} \oplus H_0(Y) \to H_0(Y_\phi) \to 0 \qquad (**)$$

In $(*)$, the homomorphisms $H_p(U\cap V)\to H_p(V)$ and $H_p(V)\to H_p(Y_\phi)$ are induced by inclusions. We should check that, for $p>0$, the homomorphism $H_p(S^{n-1})\to H_p(Y)$ in $(**)$ is induced by ϕ.

Let

$$\phi' : U-\{0\}\to U\cap V \quad \text{and} \quad \phi'' : D^n-\{0\}\to V$$

be restrictions of ϕ. The following diagram commutes, the vertical maps are inclusions:

$$\begin{tikzcd}
U-\{0\} \arrow[r, "\phi'"] \arrow[d, hook, "\alpha"'] & U\cap V \arrow[d, hook, "k"] \\
D^n-\{0\} \arrow[r, "\phi''"] & V \\
S^{n-1} \arrow[u, hook, "\beta"] \arrow[r, "\phi"'] & Y \arrow[u, hook, "j"']
\end{tikzcd}$$

Here α, β and j are homotopy equivalences. Thus α_*, β_* and j_* are isomorphisms. Since ϕ' is a homeomorphism, also ϕ'_* is an isomorphism. Thus the following diagram commutes:

$$\begin{tikzcd}
H_p(U\cap V) \arrow[r, "k_*"] & H_p(V) \\
H_p(S^{n-1}) \arrow[u, "\phi'_*\circ\alpha_*^{-1}\circ\beta_*"] \arrow[r, "\phi_*"'] & H_p(Y) \arrow[u, "j_*"']
\end{tikzcd}$$

Then the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_p(S^{n-1}) \arrow[r, "\phi_*"] \arrow[d] & H_p(Y) \arrow[r, "i_*"] \arrow[d] & H_p(Y_\phi) \arrow[r] \arrow[d] & H_{p-1}(S^{n-1}) \arrow[r] \arrow[d] & {} \\
\cdots \arrow[r] & H_p(U\cap V) \arrow[r, "k_*"'] & H_p(V) \arrow[r] & H_p(Y_\phi) \arrow[r] & H_{p-1}(U\cap V) \arrow[r] & {} \quad (*)
\end{tikzcd}$$

commutes, the vertical homomorphisms are isomorphisms.

Since $(*)$ is exact, it follows that also the top row is exact, which proves the claim. □

<u>Corollary 3.6.</u> Let Y be a compact Hausdorff space, and let Y_ϕ be the space obtained from Y by attaching an n-cell via a map ϕ, where $n \geq 2$. Then the following hold:

1) If $p \neq n, n-1$, then $H_p(Y) \cong H_p(Y_\phi)$.

2) There is an exact sequence

$$0 \longrightarrow H_n(Y) \xrightarrow{i_*} H_n(Y_\phi) \longrightarrow H_{n-1}(S^{n-1}) \xrightarrow{\phi_*} H_{n-1}(Y) \xrightarrow{\ell} H_{n-1}(Y_\phi)$$

If $n \geq 3$, then the last homomorphism is surjective.

<u>Proof.</u> 1) By Theorem 3.5, the sequence

$$\cdots \to H_p(S^{n-1}) \xrightarrow{\phi_*} H_p(Y) \xrightarrow{i_*} H_p(Y_\phi) \to H_{p-1}(S^{n-1}) \to \cdots$$

$$\cdots \to H_0(S^{n-1}) \to \mathbb{Z} \oplus H_0(Y) \to H_0(Y_\phi) \to 0 \qquad (\square)$$

is exact. If $p \neq n, n-1, 0, 1$, then $H_p(S^{n-1}) \cong 0 \cong H_{p-1}(S^{n-1})$. It follows from $(\square)$, that $H_p(Y) \cong H_p(Y_\phi)$. Attaching an n-cell, where $n \geq 2$, does not change the number of path components. Thus $H_0(Y) \cong H_0(Y_\phi)$. Let then $p = 1$. Since $p \neq n-1$, must be $n > 2$. Then $(\square)$ yields

$$0 \to H_1(Y) \xrightarrow{i_*} H_1(Y_\phi) \xrightarrow{\beta_*} \underbrace{H_0(S^{n-1})}_{\cong \mathbb{Z}} \xrightarrow{\alpha_*} \mathbb{Z} \oplus H_0(Y)$$

Here, α_* is injective (see the proof of Thm 3.5). Thus $\beta_* = 0$ and i_* is surjective. Since i_* is also injective, it follows that $H_1(Y) \cong H_1(Y_\phi)$. (Perhaps a better way to do this part is to use an M-V sequence for reduced homology in $(*)$ of Thm 3.5. Then the case $p = 1$ follows immediately.)

2) The exact sequence of Theorem 3.5 yields

$$0 \to H_n(Y) \xrightarrow{i_*} H_n(Y_f) \to H_{n-1}(S^{n-1}) \xrightarrow{d_*} H_{n-1}(Y) \xrightarrow{\delta} H_{n-1}(Y_f) \to H_{n-2}(S^{n-1}),$$

If $n \geq 3$, then $n-2 \geq 1$ and $n-2 \neq n-1$. Thus $H_{n-2}(S^{n-1}) = 0$. It follows that δ is a surjection $\square$

We next consider projective spaces:

Theorem 3.7.

1) For each $n \geq 1$, $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell. There is a disjoint union

$$\mathbb{R}P^n = e^0 \cup e^1 \cup \dots \cup e^n.$$

2) For each $n \geq 1$, $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a $2n$-cell. There is a disjoint union

$$\mathbb{C}P^n = e^0 \cup e^2 \cup \dots \cup e^{2n}.$$

3) For each $n \geq 1$, $\mathbb{H}P^n$ is obtained from $\mathbb{H}P^{n-1}$ by attaching a $4n$-cell. There is a disjoint union

$$\mathbb{H}P^n = e^0 \cup e^4 \cup \dots \cup e^{4n}.$$

proof.

1) Let $[x] = [x_1, \dots, x_{n+1}]$ be the equivalence class of $x = (x_1, \dots, x_{n+1}) \in S^n$ in $\mathbb{R}P^n$. Let

$$e = \{[x_1, \dots, x_{n+1}] \in \mathbb{R}P^n \mid x_{n+1} \neq 0\}.$$

Then $Y = \mathbb{R}P^n - e$ is an embedded copy of $\mathbb{R}P^{n-1}$.

The map

$$h: e \to \mathbb{R}^n, \ [x_1, \dots, x_{n+1}] \mapsto (x_{n+1}^{-1} x_1, \dots, x_{n+1}^{-1} x_n),$$

is a homeomorphism (check this!). Moreover, there is a homeomorphism $\mathbb{R}^n \approx D^n - S^{n-1}$. Thus e is an n-cell.

Let

$$\phi : (D^n, S^{n-1}) \to (e \cup Y, Y) = (\mathbb{R}P^n, \mathbb{R}P^{n-1})$$
$$u = (u_1, \dots, u_n) \mapsto [u_1, \dots, u_n, \sqrt{1 - \|u\|^2}\,].$$

If $u \in S^{n-1}$, then $\|u\| = 1$ and $\phi(u) = [u_1, \dots, u_n, 0] \in \mathbb{R}P^{n-1}$. Clearly, ϕ is continuous. The restriction

$$\phi| : D^n - S^{n-1} \to (e \cup Y) - Y = e$$

is a bijection. The inverse map of $\phi|$ is

$$g : e \to D^n - S^{n-1}, \ [v_1, \dots, \underbrace{v_{n+1}}_{\substack{\text{may assume} \\ v_{n+1} > 0}}] \mapsto \underbrace{\left(\frac{v_1}{\|v\|}, \dots, \frac{v_n}{\|v\|} \right)}_{\substack{\text{norm of this } < 1, \\ \text{since } v_{n+1} \neq 0}},$$

where $v = (v_1, \dots, v_{n+1})$. Then $\phi|$ is a homeomorphism (check that g is continuous). Let $f = \phi| : S^{n-1} \to Y = \mathbb{R}P^{n-1}$. It now follows from Theorem 2.11, that the map

$$D^n \amalg_f \mathbb{R}P^{n-1} \to e \cup \mathbb{R}P^{n-1} = \mathbb{R}P^n, \ [u] \mapsto (\phi \amalg id_{\mathbb{R}P^{n-1}})(u),$$

is a homeomorphism. The decomposition

$$\mathbb{R}P^n = e^0 \cup e^1 \cup \dots \cup e^n$$

can be proved by induction on n.

2 & 3: Identify complex numbers as ordered pairs of real numbers and quaternions as ordered quadruples of real numbers. The proofs are similar to the proof of Case 1. $\square$

We next calculate the homologies of complex and quaternionic projective spaces.

Theorem 3.8.

$$H_p(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & \text{if } p = 0, 2, 4, \ldots, 2n \\ 0, & \text{otherwise} \end{cases}$$

$$H_p(\mathbb{H}P^n) = \begin{cases} \mathbb{Z}, & \text{if } p = 0, 4, 8, \ldots, 4n \\ 0, & \text{otherwise} \end{cases}$$

Proof. We prove the formula for $H_p(\mathbb{C}P^n)$ by induction on $n \geq 0$. For $n = 0$, $\mathbb{C}P^0 = (\mathbb{C} - \{0\})/\sim$ is just a point. Thus $H_0(\mathbb{C}P^0) = \mathbb{Z}$ and $H_p(\mathbb{C}P^0) = 0$ for $p > 0$, and the formula holds. By an earlier exercise, $\mathbb{C}P^1 \approx S^2$. Thus $H_p(\mathbb{C}P^1) = \begin{cases} \mathbb{Z}, & \text{if } p = 0, 2, \\ 0, & \text{otherwise} \end{cases}$

and the formula holds. Assume then that the formula holds for $n \geq 1$, and consider $\mathbb{C}P^{n+1}$. By Theorem 3.4 (2), $\mathbb{C}P^{n+1}$ is obtained from $\mathbb{C}P^n$ by attaching a $2(n+1)$-cell. Since $n \geq 1$, it follows that $2(n+1) > 3$. By Corollary 3.6,

$$H_p(\mathbb{C}P^n) \cong H_p(\mathbb{C}P^{n+1}),$$

for $p \neq 2n+1, 2n+2$. By induction,

$$H_p(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & \text{if } p = 0, 2, 4, \ldots, 2n \\ 0, & \text{otherwise} \end{cases}$$

It suffices to show that $H_{2n+1}(\mathbb{C}P^{n+1}) = 0$ and $H_{2n+2}(\mathbb{C}P^{n+1}) = \mathbb{Z}$. By Corollary 3.6, there is an exact sequence

$$0 \to H_{2n+2}(\mathbb{C}P^n) \to H_{2n+2}(\mathbb{C}P^{n+1}) \to \underbrace{H_{2n+1}(S^{2n+1})}_{\cong \mathbb{Z}}$$

$$\to H_{2n+1}(\mathbb{C}P^n) \to H_{2n+1}(\mathbb{C}P^{n+1}) \to 0.$$

By induction, $H_{2n+2}(\mathbb{C}P^n) = 0 = H_{2n+1}(\mathbb{C}P^n)$. Then $H_{2n+2}(\mathbb{C}P^{n+1}) = \mathbb{Z}$ and $H_{2n+1}(\mathbb{C}P^{n+1}) = 0$. Consequently, the formula holds for $H_p(\mathbb{C}P^{n+1})$, for all p.

The quaternionic case is similar. $\square$

4. CW Complexes

<u>Definition 4.1.</u> Let X be a set and let J be an index set (J may be infinite). Let $A_j \subset X$, for every $j \in J$, and assume $X = \bigcup_{j \in J} A_j$. Assume the following conditions hold:

1) A_j is a topological space, for every $j \in J$.

2) $\forall j, k \in J$: the relative topologies on $A_j \cap A_k$ induced from A_j and A_k are the same.

3) $\forall j, k \in J$: $A_j \cap A_k$ is closed in A_j and in A_k.

The family $\{A_j \mid j \in J\}$ defines a <u>weak topology</u> on X: Let $F \subset X$. Then F is defined to be closed in X if $F \cap A_j$ is closed in A_j for all $j \in J$.

<u>Exercise</u> Check that the weak topology on X, determined by $\{A_j \mid j \in J\}$, is a topology.

It follows immediately from the definition of the weak topology, that

1) A_j is closed in X, for all $j \in J$.

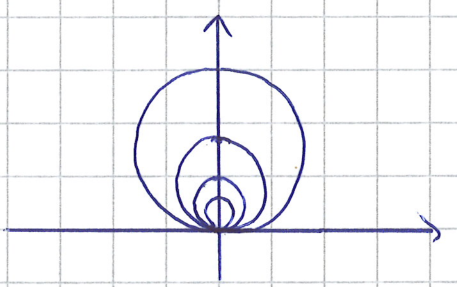
2) The relative topology on A_j induced from X equals the original topology on A_j, $\forall j \in J$.

Examples:

1) Let $\{X_j \mid j \in J\}$ be a family of topological spaces. The coproduct $X = \amalg X_j$ is the disjoint union of the X_j equipped with the weak topology determined by $\{X_j \mid j \in J\}$. Let $i \in I$. Then $X_i \cap X_i = X_i \Subset X_i$ and, for $j \neq i$, $X_i \cap X_j = \phi \Subset X_j$. Thus X_i is closed in X. Similarly, $X_i \cap X_i = X_i \Subset X_i$ and, for $j \neq i$, $X_i \cap X_j = \phi \Subset X_j$. HW $\Rightarrow$ X_i is open in X.

2) Let $\{(X_j, x_j) \mid j \in J\}$ be a family of pointed topological spaces. Assume $\{x_j\}$ is closed in X_j, for every $j \in J$. The wedge $\bigvee X_j$ is the quotient space $\amalg X_j/\sim$, when all the basepoints x_j are identified. Thus $\bigvee X_j$ is a pointed space whose basepoint is the equivalence class of the x_j. For every $j' \in J$, there is an embedding $X_{j'} \to \bigvee X_j$. Check: The quotient topology on $\bigvee X_j$ equals the weak topology determined by $\{X_j \mid j \in J\}$ when each X_j is considered as a subspace of $\bigvee X_j$.

3) Let C_n, $n \geq 1$, be the circle in $\mathbb{R}^2$ whose center is $(0, \frac{1}{n})$ and radius is $\frac{1}{n}$. Let $Y = \bigcup_n C_n$.



For every $i \geq 1$, let $S^1_i \approx S^1$, and let $X = \bigvee_i S^1_i$. We show that X and Y are not homeomorphic: For every $n \geq 1$, let $x_n \in C_n - \{(0,0)\}$. Let $F = \{x_n \mid n \geq 1\}$. Then $F \cap C_n = \{x_n\} \Subset C_n$, for every n. Thus F is a closed subset of X. Since $x_n \to (0,0)$ when $n \to \infty$, $(0,0) \in Y$ and $(0,0) \notin F$, it follows that F is not closed in Y. Thus X and Y are not homeomorphic.

Let D_n = the circle in $\mathbb{R}^2$ with center $(0,n)$ and radius n, $n \geq 1$. Let $Z = \bigcup_n D_n$. Then Z is not compact. Since Y is compact, it follows that Y and Z are not homeomorphic. Moreover, X and Z are not homeomorphic (proof: like the proof for X and Y, choose a sequence of points in Z that are contained in a compact neighborhood of the origin and converge to $(0,0)$.).

<u>Lemma 4.2.</u> Let X have the weak topology determined by a family $\{A_j \mid j \in J\}$ of subsets of X. Let Y be a topological space. Let $f : X \to Y$ be a function. Then f is continuous if and only if the restriction $f|A_j : A_j \to Y$ is continuous for every $j \in J$.

<u>proof.</u> 1) Always: f continuous $\Rightarrow$ $f|A_j$ continuous for any $A_j \subset X$.

2) Assume the restrictions $f|A_j : A_j \to Y$, $j \in J$, are continuous. Let $F \in Y$. Then

$$f^{-1}(F) \cap A_j = (f|A_j)^{-1}(F) \in A_j \quad , \forall j \in J.$$

Thus $f^{-1}(F) \in X$, which implies that f is continuous. $\square$

<u>Examples:</u> 1) Let X_j, $j \in J$, be topological spaces. Let Y be a topological space. Let $f_j : X_j \to Y$, $j \in J$, be continuous functions. Define

$$f = \amalg f_j : \amalg X_j \to Y,$$

by setting $f(x) = f_j(x)$, for all $x \in X_j$ (there is a unique j s.t. $x \in X_j$, thus f is well-defined). Then $f|X_j = f_j$ is continuous, and it follows that f is continuous.

2) Let $\{(X_j, x_j) \mid j \in J\}$ be a family of pointed spaces. (Assume $\{x_j\} \in X_j$ $\forall j \in J$.) Let (Y, y_0) be a pointed space. Let $f_j : (X_j, x_j) \to (Y, y_0)$ be continuous pointed maps, for all $j \in J$. Define

$$f = \vee f_j : (\vee X_j, e) \to (Y, y_0),$$

as follows: If $x \in \vee X_j$, $x \neq e$, then $\exists! \, X_j : x \in X_j$. Define $f(x) = f_j(x_j)$. If $x = e$, define $f(e) = y_0$. Since $f|X_j = f_j$ for all $j \in J$ and since the topology on $\vee X_j$ equals the weak topology, it follows from Lemma 4.2, that f is continuous.

<u>Definition 4.3.</u> Let X be a topological space. Assume X is a disjoint union of cells, $X = \bigcup_{e \in E} e$ (where E is an indexing set). Let $k \geq 0$.

The <u>k-skeleton</u> of X is

$$X^{(k)} = \bigcup \{ e \in E \mid \dim(e) \leq k \}.$$

Then $X^{(0)} \subset X^{(1)} \subset X^{(2)} \subset \dots$ and $X = \bigcup_{k \geq 0} X^{(k)}$.

<u>Definition 4.4.</u> A <u>CW complex</u> is an ordered triple (X, E, Φ), where X is a Hausdorff space, E is a family of cells in X, and $\Phi = \{\Phi_e \mid e \in E\}$ is a family of maps satisfying the following conditions:

1) X is the disjoint union $\bigcup \{ e \mid e \in E \}$.

2) For every k-cell $e \in E$, the map

$$\Phi_e : (D^k, S^{k-1}) \to (e \cup X^{(k-1)}, X^{(k-1)})$$

is a relative homeomorphism.