The open sets $V \in \mathcal{V}$ are called <u>coordinate neighborhoods</u>.

Notice that in a locally trivial bundle all fibers are homeomorphic to F.

<u>Theorem 16.7.</u> A locally trivial bundle $p: E \to B$ with fiber F is a Serre fibration.

<u>proof:</u> Rotman, Theorem 11.52. $\square$

<u>Theorem 16.8.</u> (Gysin Sequence)

Let $\nu: E \to B$ be a fibration with fiber S^q, where $q \geq 0$. Denote $\mathbb{Z}/2\mathbb{Z}$ by 2.

1) There is an exact sequence

$$\cdots \to H^k(B;2) \xrightarrow{\nu^*} H^k(E;2) \to H^{k-q}(B;2) \xrightarrow{\Psi_k} H^{k+1}(B;2)$$

$$\xrightarrow{\nu^*} H^{k+1}(E;2) \to H^{k+1-q}(B;2) \to \cdots$$

which begins

$$0 \to H^0(B;2) \xrightarrow{\Psi_0} H^{q+1}(B;2) \xrightarrow{\nu^*} H^1(E;2)$$

$$\to H^1(B;2) \to \cdots$$

2) Let $[e] \in H^0(B;2)$ be the unit of the cohomology ring $H^*(B;2)$. Write

$$\Omega_B = \Psi_0([e]) \in H^{q+1}(B;2),$$

Ω_B is called the <u>characteristic class</u> of the fibration. Then the map

$$\Psi_k : H^{k-q}(B;2) \to H^{k+1}(B;2), \text{ for all } k \geq 1,$$

is given by $\beta \mapsto \Omega_B \cup \beta$.

3) Let $\nu' : E' \to B'$ be another fibration with fiber S^7. Assume that the diagram

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d, "\nu'"'] & E \arrow[d, "\nu"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

commutes. Then $f^*(\Omega_B) = \Omega_{B'}$.

proof: Spanier's "Algebraic Topology" has two different proofs of this result:

1) Theorem 11 on p. 260, no spectral sequences used in the proof.

2) Theorem 2 on p. 499, the proof uses spectral sequences. $\square$

Let R be a ring (that does not necessarily contain an identity element). Let $a \in R$. Let $\{A_i \mid i \in I\}$ be the family of all ideals of R containing a. Then

$$(a) = \bigcap_{i \in I} A_i$$

is an ideal of R, it is called the ideal generated by a. Since (a) is generated by a single element, it is called a principal ideal. It can be shown that (a) consists of all elements of the form

$$ra + as + na + \sum_{i=1}^{m} r_i a s_i,$$

where $r, s, r_i, s_i \in R$, and $n \in \mathbb{Z}$.

Just some comments about the cup product that may be of some help for doing the homework:

1) There is a ring isomorphism $H^*(\bigvee_i X_i) \cong \bigoplus_i H^*(X_i)$.
(This is of help when one shows that some space is not homotopy equivalent to a wedge sum.)

2) $H^*(S^n, \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^2)$, where α is the generator of $H^n(S^n; \mathbb{Z})$:

$$H^i(S^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, n \\ 0, & \text{otherwise} \end{cases}$$

Nontrivial cup products: $\alpha \cup 1$ and $\alpha \cup \alpha$,
$\alpha \cup 1 = \alpha$, $\alpha \cup \alpha \in H^{2n}(S^n, \mathbb{Z}) = 0 \Rightarrow \alpha \cup \alpha = 0$ (i.e., $\alpha^2 = 0$).
Thus $H^*(S^n, \mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^2$

3) $H^*(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}[\beta]/(\beta^{n+1})$, where β is a generator of $H^2(\mathbb{C}P^n, \mathbb{Z})$.

4) Return to the example on p. 199:

X, Y finite type, $H_n(X)$ free abelian $\forall n \geq 0$.

$\Rightarrow$ the cross product $\times : H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$ is a ring isomorphism. The formula for $\times$ is given by the Künneth formula

$$\alpha' : \sum_{p+q=n} H^p(X) \otimes H^q(Y) \to H^n(X \times Y)$$

($\mathrm{Tor}(H^p(X), H^q(Y)) = 0 \quad \forall p, q$ since $H^p(X)$ free abelian.)

Let $X = S^1$, $Y = S^2$.

Then there are isomorphisms α':

$$H^1(S^1) \otimes H^0(S^2) \xrightarrow{\cong} H^1(S^1 \times S^2)$$

$$H^0(S^1) \otimes H^2(S^2) \xrightarrow{\cong} H^2(S^1 \times S^2)$$

$$H^1(S^1) \otimes H^2(S^2) \xrightarrow{\cong} H^3(S^1 \times S^2),$$

where each group is $\cong \mathbb{Z}$.

Let $a =$ a generator of $H^1(S^1)$

$b =$ — " — $H^2(S^2)$

$a \otimes 1 \mapsto a' =$ — " — $H^1(S^1 \times S^2)$

$1 \otimes b \mapsto b' =$ — " — $H^2(S^1 \times S^2)$

Then $(a \otimes 1)(1 \otimes b) = a \otimes b$ is a generator of $H^1(S^1) \otimes H^2(S^2)$.

$H^*(S^1 \times S^2)$: multiplication = cup product

α' ring homomorphism

$$\Rightarrow \quad a \otimes b = \underbrace{(a \otimes 1)}_{\downarrow \, a'}\underbrace{(1 \otimes b)}_{\downarrow \, b'} \mapsto a' \cup b'.$$

$H^1(S^1) \otimes H^2(S^2) \to H^3(S^1 \times S^2)$ isomorphism

$\Rightarrow$ $a' \cup b'$ generates $H^3(S^1 \times S^2)$.

Theorem 16.9. The cohomology ring $H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$ is isomorphic to the polynomial ring $(\mathbb{Z}/2\mathbb{Z})[x]$ modulo the ideal (x^{n+1}). If Ω_n is the nonzero element of $H^1(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$, then the nonzero element of $H^k(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$, $1 \le k \le n$, is Ω_n^k, the cup product of Ω_n with itself k times.

proof. Let $\nu : S^n \to \mathbb{RP}^n$ be the quotient map. Then ν is a fibration with fiber S^0. Again, denote $\mathbb{Z}/2\mathbb{Z}$ by 2. The Gysin sequence of ν is

$$\cdots \to H^k(\mathbb{RP}^n, 2) \xrightarrow{\nu^*} H^k(S^n, 2) \to H^k(\mathbb{RP}^n, 2)$$
$$\xrightarrow{\psi_k} H^{k+1}(\mathbb{RP}^n, 2) \xrightarrow{\nu^*} H^{k+1}(S^n, 2) \to H^{k+1}(\mathbb{RP}^n, 2) \to \cdots$$

Here $H^k(S^n, 2) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & k = 0, n \\ 0, & \text{otherwise.} \end{cases}$ $\quad H^k(\mathbb{RP}^n, 2) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$

Thus: $0 < k < n-1 \Rightarrow \psi_k$ is an isomorphism

$k = 0$:

$$0 \to H^0(\mathbb{RP}^n, 2) \xrightarrow{\psi_0} H^1(\mathbb{RP}^n, 2) \xrightarrow{\nu^*} H^1(S^n, 2) \to \cdots$$

$\Rightarrow \psi_0$ injective

$n > 1 \Rightarrow H^1(S^n, 2) = 0 \Rightarrow \nu^* = 0 \Rightarrow \psi_0$ surjective

covers this case $\to$ $n = 1$: check! also now $\nu^* = 0 \Rightarrow \psi_0$ surj. $\leftarrow$ just as below

$k = n-1$: $\Rightarrow \psi_k = \psi_{n-1}$ is injective

$$H^{n-1}(\mathbb{RP}^n, 2) \xrightarrow{\psi_{n-1}} H^n(\mathbb{RP}^n, 2) \xrightarrow{\nu^*} H^n(S^n, 2) \xrightarrow{\phi} H^n(\mathbb{RP}^n, 2) \xrightarrow{\psi_n} H^{n+1}(\mathbb{RP}^n, 2) \to \cdots$$

$H^{n+1}(\mathbb{RP}^n, 2) = 0 \Rightarrow \phi$ surj $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \Rightarrow \phi$ isom

$\Rightarrow \nu^* = 0$

$\Rightarrow \psi_{n-1}$ surjection

Let $\Omega_n \in H^1(\mathbb{RP}^n, 2) \cong \mathbb{Z}/2\mathbb{Z}$, $\Omega_n \neq 0$.

Let $[e] \in H^0(\mathbb{RP}^n, 2)$ be the unit of the cohomology ring $H^*(\mathbb{RP}^n, 2)$. Then ψ_0 injective $\Rightarrow \psi_0([e]) = \Omega_n$.

Gysin sequence (2): $\psi_k : H^k(\mathbb{RP}^n, 2) \to H^{k+1}(\mathbb{RP}^n, 2)$, $k \geq 1$,
$\beta \mapsto \Omega_n \cup \beta$.

By induction $\Omega_n^k = \underbrace{\Omega_n \cup \cdots \cup \Omega_n}_{k \text{ copies}} \neq 0$ for $k \leq n$.

On the other hand, $\Omega_n^{n+1} \in H^{n+1}(\mathbb{RP}^n, 2) = 0 \Rightarrow \Omega_n^{n+1} = 0$.

$\square$

Corollary 16.10. Let $n > m \geq 1$, and let $f : \mathbb{RP}^n \to \mathbb{RP}^m$ be a continuous map. Let Ω_m be the nonzero element in $H^1(\mathbb{RP}^m, \mathbb{Z}/2\mathbb{Z})$. Then $f^*(\Omega_m) = 0$.

proof. $H^1(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z})$ has two elements, 0 and Ω_n. Since $f^*(H^1(\mathbb{RP}^m, \mathbb{Z}/2\mathbb{Z})) \subset H^1(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z})$, must be $f^*(\Omega_m) = 0$ or $f^*(\Omega_m) = \Omega_n$. Assume $f^*(\Omega_m) = \Omega_n$. Then $f^*(\Omega_m^{m+1}) = (f^*\Omega_m)^{m+1} = \Omega_n^{m+1}$. However, this is impossible, since $\Omega_m^{m+1} = 0 \Rightarrow f^*(\Omega_m^{m+1}) = 0$, but $\Omega_n^{m+1} \neq 0$, since $m+1 \leq n$. $\square$

Theorem 16.11. Let $n > m \geq 1$. Then $\mathbb{RP}^m$ is not a retract of $\mathbb{RP}^n$.

proof. Let $i : \mathbb{RP}^m \hookrightarrow \mathbb{RP}^n$ be an injection. Assume there is $r : \mathbb{RP}^n \to \mathbb{RP}^m$ s.t. $ri = id : \mathbb{RP}^m \to \mathbb{RP}^m$. Then $i^*r^* : H^*(\mathbb{RP}^m, 2) \to H^*(\mathbb{RP}^m, 2)$ is the identity. Thus $r^* : H^*(\mathbb{RP}^m, 2) \to H^*(\mathbb{RP}^n, 2)$ is an injection. In particular, then $r^*(\Omega_m) \neq 0$. By Corollary 16.10, this is impossible. $\square$

The <u>usual imbedding</u> $i : \mathbb{R}P^m \to \mathbb{R}P^n$, $n \geq m \geq 1$, is the following: For $(x_0, \dots, x_m) \in S^m$, let $[x_0, \dots, x_m]$ denote the equivalence class of $(x_0, \dots, x_m)$, obtained by identifying the antipodal points. Let

$$\tilde{i} : S^m \to S^n, \; (x_0, \dots, x_m) \mapsto (x_0, \dots, x_m, 0, \dots, 0).$$

Let

$$\nu' : S^m \to \mathbb{R}P^m \quad \text{and} \quad \nu : S^n \to \mathbb{R}P^n$$

be the quotient maps. Then ν' and ν are fibrations with fiber S^0. The usual imbedding i is the imbedding induced by $\tilde{i}$, i.e., the imbedding that makes the following diagram commute:

$$\begin{tikzcd}
S^m \arrow[r, hook, "\tilde{i}"] \arrow[d, "\nu'"'] & S^n \arrow[d, "\nu"] \\
\mathbb{R}P^m \arrow[r, hook, "i"'] & \mathbb{R}P^n
\end{tikzcd}$$

Part 3 of the Gysin exact sequence $\Rightarrow$ $i^*(\Omega_n) = \Omega_m$.

<u>Definition 16.12.</u> Let $\tilde{X}$ and X be topological spaces, and let $p : \tilde{X} \to X$ be continuous. An open subset U of X is called <u>evenly covered</u> by p, if $p^{-1}(U)$ is a disjoint union of open subsets S_i of $\tilde{X}$, called <u>sheets</u>, such that the restriction $p|S_i : S_i \to U$ is a homeomorphism for every i.

Definition 16.13. Let X be a topological space. An ordered pair $(\tilde{X}, p)$ is called a <u>covering space</u> of X, if

1) $\tilde{X}$ is a path connected topological space,
2) $p : \tilde{X} \to X$ is continuous,
3) each $x \in X$ has an open neighborhood $U = U_x$ that is evenly covered by p.

Example: Let $e : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$. Let $U = S^1 - \{-1\}$. Then

$$e^{-1}(U) = \bigcup_{n \in \mathbb{Z}} \overbrace{(n - \tfrac{1}{2}, n + \tfrac{1}{2})}^{S_n}.$$

Then the S_n are open subsets in $e^{-1}(U)$, and $S_n \cap S_m = \emptyset$, for $n \neq m$. Moreover, $e| S_n \to U$ is a homeomorphism, $\forall n$. Thus U is evenly covered by the S_n.

The map p in Definition 16.13 is called the <u>covering projection</u>. An open set that is evenly covered by p is called <u>p-admissible</u> or <u>admissible</u>.

Lemma 16.14. Let $(\tilde{X}, p)$ be a covering space of X. Then p is an open continuous surjection, and hence an identification. Moreover, X is path connected.

proof. Let $x \in X$, and let $U = U_x$ be admissible. Then $p(p^{-1}U) = U$. Thus $x \in \operatorname{im} p$, and it follows that p is a surjection. Then $X = p(\tilde{X})$, where p is continuous and $\tilde{X}$ is pathconnected. It follows that X is pathconnected.

We show that p is open: Let V be open in $\tilde{X}$, and let $x \in p(V)$. Let U be an admissible open set in X, assume $x \in U$. Let $\tilde{x} \in p^{-1}(x) \cap V$. Let $\tilde{U}$ be the sheet over U that contains $\tilde{x}$. Then $\tilde{U} \cap V$ is open in $\tilde{U}$, $\tilde{x} \in \tilde{U} \cap V$, and hence $p(\tilde{U} \cap V)$ is an open subset of U containing x. It follows that

$p(V)$ is open. Consequently, p is an open map.
Since p is an open continuous surjection, it follows that p is an identification. $\square$

(By definition, a continuous surjection $f: X \to Y$ is an identification if a subset U of Y is open if and only if $f^{-1}(U)$ is open in X. It is easy to see that a continuous surjection that is either open or closed is an identification.)

<u>Example:</u> The covering projection $p: \tilde{X} \to X$ need not be closed. The exponential map $e: \mathbb{R} \to S^1$ is a covering projection, but not a closed map. For example, $A = \{ n + \frac{1}{n} \mid n \geq 3 \}$ is closed in $\mathbb{R}$. However, $e(A) = \{ e^{i 2\pi (n + \frac{1}{n})} \mid n \geq 3 \} = \{ e^{i2\pi/n} \mid n \geq 3 \}$ is not closed in S^1 ($n \to \infty \Rightarrow e^{i 2\pi/n} \to e^0 = 1 \notin A$).

<u>Theorem 16.15.</u> (Lifting Criterion) Let Y be connected and locally path connected. Let $f: (Y, y_0) \to (X, x_0)$ be continuous. Let $(\tilde{X}, p)$ be a covering space of X. Let $\tilde{x}_0 \in p^{-1}(x_0)$. Then there is a unique continuous map $\tilde{f}: (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ lifting f if and only if $f_* \pi_1(Y, y_0) \subset p_* \pi_1(\tilde{X}, \tilde{x}_0)$.

<u>proof:</u> For example, see Rotman: Theorem 10.13. $\square$

$$\begin{tikzcd}
 & (X, x_0) \\
(Y, y_0) \arrow[ur, "f"] \arrow[r, "\tilde{f}"'] & (\tilde{X}, \tilde{x}_0) \arrow[u, "p"']
\end{tikzcd}$$

Next, let's return to the case of projective spaces:

Lemma 16.16. Let $n > m \geq 1$. Let $f : \mathbb{R}P^n \to \mathbb{R}P^m$ be a continuous map. Let $\nu : S^m \to \mathbb{R}P^m$ be the quotient map (i.e., the covering projection identifying antipodal points). Then there is a continuous lifting $\tilde{f} : \mathbb{R}P^n \to S^m$ (i.e., $\nu\tilde{f} = f$).

proof: Theorem 16.15 $\Rightarrow$ it suffices to prove that the induced map $f_* : \pi_1(\mathbb{R}P^n, *) \to \pi_1(\mathbb{R}P^m, *)$ is the constant map zero.

1. Let $m = 1$. Then $\mathbb{R}P^m = \mathbb{R}P^1 = S^1$. Thus $\pi_1(\mathbb{R}P^1, *) \cong \mathbb{Z}$.
$n > m \Rightarrow 0 < 1 < n$ and 1 is odd $\Rightarrow H_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$.
(actually, we do not need H_1 here but π_1)

$\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$ for $n > 1$:

The group $\mathbb{Z}/2\mathbb{Z}$ acts on S^n. This means that we can consider the elements of $\mathbb{Z}/2\mathbb{Z}$ as a group of homeomorphisms of S^2, and that these homeomorphisms form a group isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Thus: the identity element of $\mathbb{Z}/2\mathbb{Z}$ corresponds to the identity map S^n, the other element of $\mathbb{Z}/2\mathbb{Z}$ corresponds to the antipodal map $g : S^n \to S^n$, $x \mapsto -x$. Then $g(x) \neq x \ \forall x \in S^n$, which means that the action is free.
Since $n > 1$, it follows that S^n is simply-connected. $\mathbb{R}P^n$ can now be thought as the quotient $S^n/\mathbb{Z}/2\mathbb{Z}$ of S^n by the free action of $\mathbb{Z}/2\mathbb{Z}$. It follows from the properties of covering spaces, that $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$.

Thus we have
$$f_* : \begin{array}{ccc} \pi_1(\mathbb{R}P^n) & \to & \pi_1(\mathbb{R}P^1) \\ \wr\wr & & \wr\wr \\ \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z} \end{array},$$

and it follows that f_* must be the zero map.

2. Let $m \geq 2$: Let f^* be the induced map in cohomology. Corollary 16.10 $\Rightarrow f^*(\Omega_m) = 0$. Let

$$i : \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^n$$

be the usual embedding. Then

$$0 = i^* f^*(\Omega_m) = (f \circ i)^*(\Omega_m).$$

Let also $j : \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^m$ be the usual embedding.

Exercise: Show that $j^*(\Omega_m) \neq 0$.

It follows that $j, f \circ i : \mathbb{R}P^1 \to \mathbb{R}P^m$ are not homotopic. $\mathbb{R}P^1 = S^1 \Rightarrow j$ and $f \circ i$ represent elements in $\pi_1(\mathbb{R}P^m) = \mathbb{Z}/2\mathbb{Z}$ (since $m > 1$). Now $j^*(\Omega_m) \neq 0$ $\Rightarrow j$ is not nullhomotopic. $\Rightarrow f \circ i$ is nullhomotopic.

Exercise: Let $i : \mathbb{R}P^1 \to \mathbb{R}P^n$ be the usual embedding, $n \geq 2$. Let $\nu' : S^1 \to \mathbb{R}P^1$ be the quotient map. Show that $[i \circ \nu']$ is the nontrivial element in $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}$.

Let $\nu' : S^1 \to \mathbb{R}P^1$ be the quotient map. Exercise above $\Rightarrow [i \circ \nu']$ is the nontrivial element of $\pi_1(\mathbb{R}P^n)$. Thus $f_*[i \circ \nu'] = [f \circ i \circ \nu'] = 0$ since $f \circ i$ is nullhomotopic. $\square$

Theorem 16.17. Let $n > m \geq 1$. Then there is no continuous map $g : S^n \to S^m$ satisfying $g(-x) = -g(x)$ for all $x \in S^n$.

<u>proof.</u> Assume there is such a map $g: S^n \to S^m$. Then g induces a continuous map that makes the following diagram commute:

$$\begin{tikzcd} S^n \arrow[r, "g"] \arrow[d, "p'"'] & S^m \arrow[d, "p"] \\ \mathbb{R}^n \arrow[r, "f"'] & \mathbb{R}P^m \end{tikzcd}$$

p', p : the quotient maps

(*)

Lemma 16.16 $\Rightarrow$ there is a lifting $\tilde{f}: \mathbb{R}P^n \to S^m$ with $p\tilde{f} = f$. Consider the diagram

$$\begin{tikzcd} & S^m \arrow[d, "p"] \\ S^n \arrow[ur, dashed] \arrow[r, "fp'"'] & \mathbb{R}P^m \end{tikzcd}$$

(*) commutes $\Rightarrow$ g is a lifting of fp'. Also $\tilde{f}p'$ is a lifting of fp':

$$pg = fp' = p\tilde{f}p'.$$

Let $x_0 \in S^n$. Then $p(g(x_0)) = p(-g(x_0))$, thus the fiber over $p(g(x_0))$ is $\{g(x_0), -g(x_0)\}$. Now

$$p\tilde{f}p'(x_0) = p(g(x_0))$$

$\Rightarrow$ $\tilde{f}p'(x_0) = g(x_0)$ or $\tilde{f}p'(x_0) = -g(x_0)$.

If $\tilde{f}p'(x_0) = -g(x_0)$, then

$$\tilde{f}p'(-x_0) = \tilde{f}p'(x_0) = -g(x_0) = g(-x_0)$$

by the assumption about g.

Thus, in each case there is a point in S^n at which the liftings g and $\tilde{f}\nu'$ obtain the same value. Then (see Lemma 10.3 in Rotman's book), it follows from the uniqueness of liftings that $g = \tilde{f}\nu'$. This is a contradiction: Let $x \in S^n$. Then $\nu'(-x) = \nu'(x) \Rightarrow \tilde{f}\nu'(-x) = \tilde{f}\nu'(x)$. However, $g(-x) = -g(x)$ for all $x \in S^n$. $\square$

<u>Corollary 16.18.</u> (Borsuk-Ulam) Let $n \geq 1$, and let $f: S^n \to \mathbb{R}^n$ be continuous. Then there is $x \in S^n$ with $f(x) = f(-x)$.

<u>proof.</u> Let $n=1$, and let $f: S^1 \to \mathbb{R}^1$ be continuous. S^1 compact $\Rightarrow \exists\, a \in S^1 : f(x) \leq f(a) \;\forall\, x \in S^1$. $\Rightarrow f(a) - f(-a) \geq 0$. If $f(a) - f(-a) = 0$, then $f(a) = f(-a)$. Assume $f(a) - f(-a) > 0$. Then $f(-a) - f(a) = -(f(a) - f(-a)) < 0$. Let $g: S^1 \to \mathbb{R}$, $g(x) = f(x) - f(-x)$. Then g is continuous, $g(a) > 0$ and $g(-a) < 0$. S^1 connected $\Rightarrow g(S^1) \subset \mathbb{R}$ is connected. But then $0 \in g(S^1)$ $\Rightarrow \exists\, b \in S^1 : g(b) = 0 \Rightarrow f(b) = f(-b)$.

Let then $n > 1$. Assume there is no $x \in S^n$ with $f(x) = f(-x)$. Then

$$g: S^n \to S^{n-1}, \quad x \mapsto \frac{f(x) - f(-x)}{\| f(x) - f(-x) \|},$$

is a well-defined continuous map, and $g(-x) = -g(x) \;\forall\, g \in S^n$. This contradicts Theorem 16.17. $\square$