Theorem 15.13. (Alexander – Whitney)

The map $\zeta : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$ defined by

$$\zeta_n(\sigma) = \sum_{i+j=n} \sigma'\lambda_i \otimes \sigma''\mu_j ,$$

where

$$\sigma : \Delta_n \to X \times Y$$
$$\sigma' = \pi'\sigma \ , \quad \sigma'' = \pi''\sigma$$

$\pi' : X \times Y \to X$, $\pi'' : X \times Y \to Y$ are the projections,

is a natural chain equivalence over $\zeta_0 : (x,y) \mapsto x \otimes y$.

proof.

1. Naturality: Let $f : X \to X'$ and $g : Y \to Y'$ be continuous maps. Then the following diagram commutes:

$$\begin{tikzcd}
S_*(X \times Y) \arrow[r, "\zeta"] \arrow[d, "(f\times g)_\#"'] & S_*(X) \otimes S_*(Y) \arrow[d, "f_\# \otimes g_\#"] \\
S_*(X' \times Y') \arrow[r, "\zeta"'] & S_*(X') \otimes S_*(Y')
\end{tikzcd}$$

$$\begin{tikzcd}
\sigma \arrow[r, mapsto] \arrow[d, mapsto] & \sum_{i+j=n} \sigma'\lambda_i \otimes \sigma''\mu_j \arrow[d, mapsto] \\
(f\times g)_\# \sigma \arrow[dr, mapsto] & \sum_{i+j=n} f_\#\sigma'\lambda_i \otimes g_\#\sigma''\mu_j \arrow[d, equal] \\
& \sum_{i+j=n} \underbrace{\pi'((f\times g)_\#\sigma)}_{f_\#\pi'\sigma = f_\#\sigma'}\lambda_i \otimes \underbrace{\pi''((f\times g)_\#\sigma)}_{g_\#\pi''\sigma = g_\#\sigma''}\mu_j
\end{tikzcd}$$

2. ξ is a chain map:

Let $d: \Delta^n \to \Delta^n \times \Delta^n$ be the diagonal map, and let $\sigma: \Delta^n \to X \times Y$ be an n-simplex. Then

$$\sigma = (\sigma' \times \sigma'')d.$$

Assume $D_n \xi_n(d) = \xi_{n-1}\partial(d)$. $(*)$ We show that this implies that

$$D\xi(\sigma) = \xi\partial(\sigma),$$

i.e. that ξ is a chain map:

$$\begin{aligned}
D\xi(\sigma) &= D\xi((\sigma' \times \sigma'')d) \\
&= D\xi(\sigma' \times \sigma'')_\#(d) \\
&= D(\sigma'_\# \otimes \sigma''_\#)\xi(d) && \text{(naturality)} \\
&= (\sigma'_\# \otimes \sigma''_\#)D\xi(d) && (\sigma'_\# \otimes \sigma''_\# \text{ is a chain map}) \\
&= (\sigma'_\# \otimes \sigma''_\#)\xi\partial(d) && (*) \\
&= \xi(\sigma' \times \sigma'')_\#\partial(d) && \text{(naturality)} \\
&= \xi\partial(\sigma' \times \sigma'')_\#(d) && ((\sigma' \times \sigma'')_\# \text{ is a chain map}) \\
&= \xi\partial((\sigma' \times \sigma'')d) \\
&= \xi\partial(\sigma)
\end{aligned}$$

3. Prove $(*)$: $D\xi(d) = \xi\partial(d)$:

Recall: For $i = 0, 1, \ldots, n$, let e_i be the point in $\mathbb{R}^{n+1}$ whose cartesian coordinates are all 0 except for 1 in the $(i+1)$st position. Then $\Delta_n = [e_0, e_1, \ldots, e_n]$ = the set of all convex combinations $x = \sum t_i e_i$ ($\sum t_i = 1$, $t_i \geq 0 \ \forall i$). An affine map $T: \Delta_i \to \Delta_n$ is a map that satisfies $T(\sum t_j e_j) = \sum t_j T(e_j)$, whenever $\sum t_j = 1$.

Let $\alpha: \Delta^i \to \Delta^n$ be an affine map.
Assume $\alpha(e_k) \in \{e_0, \ldots, e_n\}$ $\forall k$.
Denote α by $(\alpha(e_0), \alpha(e_1), \ldots, \alpha(e_i))$.

Then $\quad \lambda_i : \Delta_i \to \Delta_n, \ (t_0, \dots, t_i) \mapsto (t_0, \dots, t_i, \overbrace{0, \dots, 0}^{n-i})$

$\quad \mu_{n-i} : \Delta_{n-i} \to \Delta_n, \ (t_0, \dots, t_{n-i}) \mapsto (\underbrace{0, \dots, 0}_{i}, t_0, \dots, t_{n-i}),$

$\Rightarrow \quad \lambda_i = (e_0, \dots, e_i)$ and $\mu_{n-i} = (e_i, \dots, e_n)$.

Now,

$$D\zeta(d) = D \sum_{i+j=n} d'\lambda_i \otimes d''\mu_j$$

$$= D \sum_i \lambda_i \otimes \mu_{n-i} \quad , \quad d : \Delta_n \to \Delta_n \times \Delta_n \ , \ d' = \pi' d = id_{\Delta_n} = d''$$

$$= D \sum_i (e_0, \dots, e_i) \otimes (e_i, \dots, e_n)$$

$$= \sum_i \left[\partial(e_0, \dots, e_i) \otimes (e_i, \dots, e_n) + (-1)^i (e_0, \dots, e_i) \otimes \partial(e_i, \dots, e_n) \right]$$

$$= \sum_{i=0}^{n} \sum_{j \leq i} (-1)^j (e_0, \dots, \hat{e}_j, \dots, e_i) \otimes (e_i, \dots, e_n)$$

$$+ \sum_{i=0}^{n} \sum_{j \geq i} (-1)^j (e_0, \dots, e_i) \otimes (e_i, \dots, \hat{e}_j, \dots, e_n)$$

The second sum above:

$$\partial(e_i, \dots, e_n) = \sum_{k=0}^{n-i} (-1)^k (e_i, \dots, \hat{e}_{i+k}, \dots, e_n)$$

$$= \sum_{j=i}^{n} (-1)^{j-i} (e_i, \dots, \hat{e}_j, \dots, e_n) \qquad j = i+k \Rightarrow k = j-i$$

The portion of the first sum with $j = i$:

$$\sum_{i=0}^{n} (-1)^i (e_0, \dots, e_{i-1}) \otimes (e_i, \dots, e_n)$$

in fact, start indexing from $i=1$

cancels the portion of the second sum with $j=i$:

$$\sum_{i=0}^{n} (-1)^i (e_0, \dots, e_i) \otimes (e_{i+1}, \dots, e_n)$$

and here the indexing ends with $i = n-1$.

Thus
$$D\xi(d) = \sum_{j<i} (-1)^j (e_0, \dots, \hat{e}_j, \dots, e_i) \otimes (e_i, \dots, e_n)$$
$$+ \sum_{j>i} (-1)^j (e_0, \dots, e_i) \otimes (e_i, \dots, \hat{e}_j, \dots, e_n).$$

On the other hand,
$$\xi_{n-1}(\partial d) = \sum_{i=0}^{n-1} (\partial d)' \lambda_i \otimes (\partial d)'' \mu_{n-1-i}.$$

Let $\pi' : \Delta_n \times \Delta_n \to \Delta_n$ be the projection on the first factor. Then
$$(\partial d)' = \pi'_\#(\partial d).$$

$\pi'_\#$ is a chain map

$\Rightarrow \quad \pi'_\#(\partial d) = \partial \pi'_\#(d) = \partial(\pi' d) = \partial(\delta^n)$,

where $\delta^n : \Delta_n \to \Delta_n$ is the identity map.

Similarly, $\quad (\partial d)'' = \partial(\delta^n)$.

Let $\varepsilon_j : \Delta_{n-1} \to \Delta_n$, $(t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1})$.

Check: $\varepsilon'_j = \varepsilon_j = \varepsilon''_j$.

Then, for every j,
$$\xi(\varepsilon_j) = \sum_i \varepsilon_j \lambda_i \otimes \varepsilon_j \mu_{n-1-i}$$

(earlier lemma $\Rightarrow$) $$= \sum_{i<j} \lambda_i \otimes \mu_{n-i} \varepsilon_{j-i-1} + \sum_{i>j} \lambda_{i+1} \varepsilon_j \otimes \mu_{n-i}$$

$$= \sum_{i<j} (e_0, \dots, e_i) \otimes (e_i, \dots, \hat{e}_j, \dots, e_n)$$
$$+ \sum_{i>j} (e_0, \dots, \hat{e}_j, \dots, e_i) \otimes (e_i, \dots, e_n)$$

Therefore,

$$\zeta(\partial d) = \sum(-1)^j \zeta(\varepsilon_j) = D\zeta(d). \quad \square$$

Definition 15.14. Define

$$\pi : S^*(X,R) \otimes S^*(Y,R) \to \mathrm{Hom}(S_*(X) \otimes S_*(Y), R)$$

as follows: Let $\varphi \in S^n(X,R)$, and let $\theta \in S^m(Y,R)$. Define

$$F : S^*(X,R) \times S^*(Y,R) \to \mathrm{Hom}(S_*(X) \otimes S_*(Y), R)$$
$$(\varphi, \theta) \mapsto \varphi \otimes \theta,$$

where

$$(\sigma_i \otimes \tau_j, \varphi \otimes \theta) = \begin{cases} (\sigma_i, \varphi)(\tau_j, \theta), & \text{if } i = n, j = m \\ 0 & , \text{otherwise.} \end{cases}$$

F bilinear $\Rightarrow$ it induces a homomorphism π on the tensor product.

Definition 15.15. Let

$$\pi : S^*(X,R) \otimes S^*(Y,R) \to \mathrm{Hom}(S_*(X) \otimes S_*(Y), R)$$

as in Def. 15.14. Let

$$\zeta : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$$

be the Alexander-Whitney map. The (external) cross product is the map

$$\zeta^\# \pi : S^*(X,R) \otimes S^*(Y,R) \to \mathrm{Hom}(S_*(X \times Y), R) = S^*(X \times Y, R).$$

Let $\varphi \in S^n(X,R)$, and let $\theta \in S^m(Y,R)$. The cross-product of φ and θ is

$$\varphi \times \theta \in S^{n+m}(X \times Y, R).$$

The cross product can also be considered as a map in cohomology

$$H^*(X,R) \otimes H^*(Y,R) \to H^*(X\times Y, R),$$

in which case it becomes the map α' in Künneth Formula:

$$\alpha' : [\varphi_i] \otimes [\theta_j] \mapsto [\xi^\#(\varphi_i \otimes \theta_j)].$$

<u>Theorem 15.16.</u> Let $d : X \to X\times X$, $x \mapsto (x,x)$, be the diagonal map. Then the cup product $\cup$ on $S^*(X,R)$ equals the composite $d^\# \xi^\# \Pi$:

$$S^*(X,R) \otimes S^*(X,R) \xrightarrow{\Pi} \mathrm{Hom}(S_*(X) \otimes S_*(X), R)$$
$$\xrightarrow{\xi^\#} S^*(X\times X, R)$$
$$\xrightarrow{d^\#} S^*(X,R).$$

<u>proof.</u> Let $\varphi \in S^n(X,R)$, $\theta \in S^m(X,R)$. Let σ be an $(n+m)$-simplex in X. Then

$$\begin{aligned}(\sigma, d^\# \xi^\# \Pi(\varphi,\theta)) &= (\sigma, d^\# \xi^\#(\varphi\otimes\theta)) \\ &= (d\sigma, \xi^\#(\varphi\otimes\theta)) \\ &= (\xi d\sigma, \varphi\otimes\theta),\end{aligned}$$

where

$$\xi(d\sigma) = \sum_{i=0}^{n+m} (d\sigma)'\lambda_i \otimes (d\sigma)''\mu_{n+m-i},$$

and

$$(d\sigma)' = \Pi' d\sigma, \quad (d\sigma)'' = \Pi''(d\sigma),$$

where $\Pi', \Pi'' : X\times X \to X$ are the projections.

$d : X \to X \times X$ is the diagonal map $\Rightarrow \pi' d = \pi'' d = id_X$. Then

$$\xi(d\sigma) = \sum_{i=0}^{n+m} \sigma\lambda_i \otimes \sigma\rho_{n+m-i}.$$

Now, $\varphi \otimes \theta$ is the zero map outside $S_n(X) \otimes S_m(X)$. Then

$$\begin{aligned}
(\xi(d\sigma), \varphi \otimes \theta) &= \left(\sum_{i=0}^{n+m} \sigma\lambda_i \otimes \sigma\rho_{n+m-i}, \varphi \otimes \theta \right) \\
&= (\sigma\lambda_n \otimes \sigma\rho_m, \varphi \otimes \theta) \\
&= (\sigma\lambda_n, \varphi)(\sigma\rho_m, \theta) \\
&= (\sigma, \varphi \cup \theta). \quad \square
\end{aligned}$$

<u>Corollary 15.17.</u> If $\varphi \in S^n(X, R)$, $\theta \in S^m(X, R)$, then

$$\varphi \cup \theta = d^{\#}(\varphi \times \theta).$$

<u>proof.</u> Follows immediately from Theorem 15.16 and from the definition of the cross product. $\square$

<u>Lemma 15.18.</u> Let (S_*, ∂) be a nonnegative chain complex. Then $T : S_* \otimes S_* \to S_* \otimes S_*$ defined by

$$T(\alpha \otimes \beta) = (-1)^{pq} \beta \otimes \alpha,$$

where $\alpha \in S_p$, $\beta \in S_q$, is a chain equivalence.

<u>proof.</u> Let D be the differentiation on $S_* \otimes S_*$:
$D_{i+j}(\alpha \otimes \beta) = \partial\alpha \otimes \beta + (-1)^i \alpha \otimes \partial\beta$, if $\alpha \in S_i(X)$, $\beta \in S_j(X)$.
Then

$$\begin{aligned}
DT(\alpha \otimes \beta) &= D\left((-1)^{pq} \beta \otimes \alpha\right) \\
&= (-1)^{pq} \left(\partial\beta \otimes \alpha + (-1)^q \beta \otimes \partial\alpha \right) \\
&= (-1)^{pq} \partial\beta \otimes \alpha + (-1)^{pq+q} \beta \otimes \partial\alpha.
\end{aligned}$$

On the other hand,

$$\begin{aligned} TD(\alpha \otimes \beta) &= T(\partial\alpha \otimes \beta + (-1)^p \alpha \otimes \partial\beta) \\ &= (-1)^{(p-1)q} \beta \otimes \partial\alpha + (-1)^{p+p(q-1)} \partial\beta \otimes \alpha \\ &= (-1)^{pq-q} \beta \otimes \partial\alpha + (-1)^{pq} \partial\beta \otimes \alpha \\ &= DT(\alpha \otimes \beta), \quad \text{since } (-1)^q = (-1)^{-q} \Rightarrow (-1)^{pq-q} = (-1)^{pq+q}. \end{aligned}$$

It follows that T is a chain map.
Check: T is a natural isomorphism. □

Theorem 15.19. (Anticommutativity)

Let $[\varphi] \in H^p(X;R)$, and let $[\theta] \in H^q(X;R)$. Then

$$[\varphi] \cup [\theta] = (-1)^{pq} [\theta] \cup [\varphi].$$

proof. Let $d: X \to X \times X$ be the diagonal map, and let

$$\xi: S_*(X \times X) \to S_*(X) \otimes S_*(X),$$

$$\xi_n(\sigma) = \sum_{i+j=n} \sigma'\lambda_i \otimes \sigma''\mu_j$$

be the Alexander-Whitney map. Let T be the map defined in Lemma 15.18. Then

$$\xi d_\#, \ T\xi d_\# : S_*(X) \to S_*(X) \otimes S_*(X)$$

are natural chain maps over

$$\xi_0 d_{\#0} : x \mapsto x \otimes x.$$

Here $\xi_0 d_{\#0} : S_0(X) \to S_0(X) \otimes S_0(X)$ is a natural equivalence. Therefore, by the Acyclic Models Theorem, $\xi d_\#$ and $T\xi d_\#$ are naturally chain equivalent.

Apply the functor $\mathrm{Hom}(\ ,R) \Rightarrow \xi d^\#$ and $J\xi d^\#$ induce the same homomorphism in cohomology: $[d^\# \xi^\#(\ )] = [d^\# \xi^\# J^\#(\ )]$. The claim follows, since the cup product equals $d^\# \xi^\# \pi$. $\square$

16. Computations and Applications

The following lemma is useful in computing cohomology rings:

Lemma 16.1.

1) Let R and S be rings. Then $R \otimes S$ is a ring, the multiplication is defined by

$$(r \otimes s)(r' \otimes s') = rr' \otimes ss', \quad \text{where } r, r' \in R,\ s, s' \in S.$$

2) Let R and S be graded rings. Then $R \otimes S$ is a graded ring. Multiplication on $R \otimes S$ is defined by

$$(r_i \otimes s_j)(r'_p \otimes s'_q) = (-1)^{jp} r_i r'_p \otimes s_j s'_q,$$

where $r_i \in R_i$, $r'_p \in R_p$, $s_j \in S_j$, $s'_q \in S_q$, and

$$(R \otimes S)_n = \sum_{i+j=n} R_i \otimes S_j.$$

proof.

1) Let μ be the multiplication on R. Then μ is a bilinear function, and it induces $\tilde{\mu}: R \otimes R \to R$, $r \otimes r' \mapsto \mu(r, r') = rr'$.

$$\begin{tikzcd} R \times R \arrow[r, "p"] \arrow[dr, "\mu"'] & R \otimes R \arrow[d, "\tilde{\mu}"] \\ & R \end{tikzcd}$$

p = the quotient map

Similarly, the multiplication $\sigma: S\times S \to S$ induces $\tilde{\sigma}: S\otimes S \to S$. Thus

$$(R\otimes S)\times(R\otimes S) \xrightarrow{\otimes} (R\otimes S)\otimes(R\otimes S) \xrightarrow{id\otimes\varphi\otimes id} (R\otimes R)\otimes(S\otimes S) \xrightarrow{\tilde{\mu}\otimes\tilde{\sigma}} R\otimes S$$
$$(r\otimes s, r'\otimes s') \mapsto (r\otimes s)\otimes(r'\otimes s) \mapsto (r\otimes r')\otimes(s\otimes s') \mapsto rr'\otimes ss'$$

is well-defined. (Here $\varphi: S\otimes R \to R\otimes S$, $s\otimes r \mapsto r\otimes s$.)

Check: This multiplication makes $R\otimes S$ a ring.

2) The term $(-1)^{pq}$ appears here so that the multiplication becomes anticommutative. $\square$

Theorem 16.2.

1) Let X and Y be topological spaces. Then the cross product $H^*(X)\otimes H^*(Y) \to H^*(X\times Y)$ is a homomorphism of graded rings.

2) Let X and Y be topological spaces of finite type (for example, compact CW complexes). Assume $H_n(X)$ is a free abelian group, for all $n\geq 0$. Then the cross product is an isomorphism.

proof. By Künneth formula, there is an exact sequence

$$0 \to \sum_{i+j=n} H^i(X)\otimes H^j(Y) \xrightarrow{\alpha'} H^n(X\times Y) \to \sum_{p+q=n+1} \mathrm{Tor}(H^p(X), H^q(Y)) \to 0,$$

where $\alpha': [\varphi_i]\otimes[\theta_j] \mapsto [\xi^\#(\varphi_i\otimes\theta_j)]$ is the cross product.

2 follows from 1: Dual coefficient theorem:

$$H^n(X) \cong \mathrm{Hom}(H_n(X), \mathbb{Z}) \oplus \mathrm{Ext}(H_{n-1}(X), \mathbb{Z})$$

$H_{n-1}(X)$ free abelian $\Rightarrow$ $\mathrm{Ext}(H_{n-1}(X), \mathbb{Z}) = 0$, by Ext 2.

$H_n(X)$ free abelian $\Rightarrow$ $\mathrm{Hom}(H_n(X), \mathbb{Z})$ free abelian.

Thus $H^n(X)$ is free abelian, for all $n \geq 0$.

$\Rightarrow$ $\mathrm{Tor}(H^p(X), H^q(Y)) = 0$ $\forall p$, by Tor 2.

Then Künneth formula $\Rightarrow$ α' is an isomorphism.

1) To prove the first part of the theorem, we need the following exercise:

<u>Exercise:</u> Let $f: X \to X'$ and $g: Y \to Y'$ be continuous functions. Let $[\varphi] \in H^p(X', R)$, and let $[\theta] \in H^q(Y', R)$. Then $(f \times g)^*([\varphi] \times [\theta]) = f^*[\varphi] \times g^*[\theta]$.

Return to the proof of 1:

Let $\varphi \in Z^n(X, R)$, $\varphi' \in Z^p(X, R)$, $\theta \in Z^m(Y, R)$, $\theta' \in Z^q(Y, R)$, and let $U = [\varphi]$, $U' = [\varphi']$, $V = [\theta]$, $V' = [\theta']$. We show that

$$\alpha'((U \otimes V)(U' \otimes V')) = \alpha'(U \otimes V) \cup \alpha'(U' \otimes V').$$

Here

$$(U \otimes V)(U' \otimes V') = (-1)^{mp} (U \cup U') \otimes (V \cup V'), \text{ by definition.}$$

Since $\alpha'(U \otimes V) = U \times V$, it suffices to prove that

$$(-1)^{mp} (U \cup U') \times (V \cup V') = (U \times V) \cup (U' \times V').$$

Let $d_X: X \to X \times X$, $x \mapsto (x, x)$, and let

$$T: X \times Y \to Y \times X, \quad (x, y) \mapsto (y, x).$$

Then the following diagram commutes:

$$
\begin{tikzcd}
X \times Y \arrow[rr, "d_x \times d_y"] \arrow[dr, "d_{X\times Y}"'] & & X \times (X \times Y) \times Y \arrow[dl, "id \times T \times id"] \\
& X \times (Y \times X) \times Y &
\end{tikzcd}
$$

and it induces the following commotative diagram:

$$
\begin{tikzcd}
H^*(X \times Y) & & H^*(X \times (X \times Y) \times Y) \arrow[ll, "(d_x \times d_Y)^*"'] \\
& H^*(X \times (Y \times X) \times Y) \arrow[ul, "d^*_{X\times Y}"] \arrow[ur, "id^* \times T^* \times id^*"'] &
\end{tikzcd}
$$

Notice the following:

if $\varphi : S_*(X) \to R$ cocycle
$\quad \theta : S_*(X) \to R$, —"—
then $\varphi \otimes \theta \in S^*(X, R) \otimes S^*(X, R)$
$= Hom(S_*(X), R) \otimes Hom(S_*(Y), R)$,
and
$$\pi(\varphi \otimes \theta) \in Hom(S_*(X) \otimes S_*(X), R).$$

To be precice, we should write the map α' in the Künneth formula as

$$\alpha' : \sum_{i+j=n} H^i(X) \otimes H^j(X) \to H^n(X \times X)$$
$$[\varphi_i] \otimes [\theta_j] \mapsto [\xi^\# \pi(\varphi_i \otimes \theta_j)]$$

Then
$$
\begin{aligned}
d^*([\varphi] \times [\theta]) &= d^*(\alpha'([\varphi] \otimes [\theta])) \\
&= d^*[\xi^\# \pi(\varphi \otimes \theta)] \\
&= [d^\# \xi^\# \pi(\varphi \otimes \theta)] \\
&= [\varphi \cup \theta] \text{ by Thm 15.16} \\
&= [\varphi] \cup [\theta] \qquad (*)
\end{aligned}
$$

Again, return to the proof of I:

$$
\begin{aligned}
(U\times V)\cup(U'\times V') &= d^*_{X\times Y}\big((U\times V)\times(U'\times V')\big) && \text{by } (*)\\
&= (d_X\times d_Y)^*(id\times T\times id)^*\big((U\times V)\times(U'\times V')\big) && (\text{diagram commutes})\\
&= (d_X\times d_Y)^*\big(U\times T^*(V,U')\times V'\big) && (\text{Exercise})\\
&= (-1)^{mp}(d_X\times d_Y)^*(U\times U'\times V\times V') && (\text{Lemma } 15.18)\\
&= (-1)^{mp}\, d_X^*(U\times U')\times d_Y^*(V\times V')\\
&= (-1)^{mp}(U\cup U')\times(V\cup V') && (*)
\end{aligned}
$$

$\square$

Notice:

Cross-product:

$$S^*(X,R)\otimes S^*(Y,R)\longrightarrow S^*(X\times Y,R)$$
$$\varphi\otimes\theta\longmapsto\varphi\times\theta$$

$$H^*(X,R)\otimes H^*(Y,R)\longrightarrow H^*(X\times Y,R)$$
$$[\varphi]\otimes[\theta]\longmapsto[\varphi\times\theta]$$

$d: X\to X\times X$ diagonal map

Cup-product:

$$S^*(X,R)\otimes S^*(X,R)\xrightarrow{\times} S^*(X\times X,R)\xrightarrow{d^\#} S^*(X,R)\,,\quad \varphi\cup\theta=d^\#(\varphi\times\theta)$$

$$H^*(X,R)\otimes H^*(X,R)\xrightarrow{\times} H^*(X\times X,R)\xrightarrow{d^*} H^*(X,R)$$

$$
\begin{aligned}
[\varphi]\cup[\theta]=[\varphi\cup\theta]=[d^\#(\varphi\times\theta)] &= d^*[\varphi\times\theta]\\
&= d^*([\varphi]\times[\theta])
\end{aligned}
$$

$$X \xrightarrow{d} X \times X \xrightarrow{p_1} X \quad , \; p_1 : X \times X \to X, \; (x,y) \mapsto x,$$

$$p_1 \circ d = id_X \quad \Rightarrow \quad d^* \circ p_1^* = (p_1 \circ d)^* = id_X^* = id : H^*(X) \to H^*(X)$$

$\Rightarrow$ d^* is a surjection

<u>Example:</u> The cohomology ring is a finer invariant than the homology or cohomology groups. Consider the spaces $S^1 \times S^2$ and $S^1 \vee S^2 \vee S^3$.

Künneth formula: $H_n(S^1 \times S^2) \cong \sum_{i+j=n} H_i(S^1) \otimes H_j(S^2)$

$$\oplus \sum_{p+q=n-1} Tor(H_p(S^1), H_q(S^2))$$

$$\Rightarrow H_n(S^1 \times S^2) = \begin{cases} \mathbb{Z}, & n = 0,1,2,3 \\ 0, & \text{otherwise} \end{cases} = H_n(S^1 \vee S^2 \vee S^3)$$

Dual Universal Coefficients:

$$H^n(X;G) \cong Hom(H_n(X), G) \oplus \underbrace{Ext(H_{n-1}(X), G)}_{0, \text{ if } H_{n-1}(X) \text{ is free abelian}}$$

$$Hom(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$$

$$\Rightarrow H^n(S^1 \times S^2) = H^n(S^1 \vee S^2 \vee S^3) = \begin{cases} \mathbb{Z}, & n = 0,1,2,3 \\ 0, & \text{otherwise} \end{cases}$$

Now,

$$\cup : H^1(S^1 \times S^2) \otimes H^2(S^1 \times S^2) \xrightarrow{\times} H^3((S^1 \times S^2) \times (S^1 \times S^2)) \xrightarrow{d^*} H^3(S^1 \times S^2).$$

$S^1 \times S^2$ is of finite type $\overset{16.2}{\Rightarrow}$ $\times$ is an isomorphism.
d^* is a surjection.

Let a = the generator of $H^1(S^1 \times S^2)$
b = the generator of $H^2(S^1 \times S^2)$.

Then $\times$ isom., d^* surj $\Rightarrow$ $\cup = d^* \circ \times$ is surj
$\Rightarrow$ $a \cup b$ is the generator of $H^3(S^1 \times S^2)$.

Let then $d : S^1 \vee S^2 \vee S^3 \to S^1 \vee S^2$ be the identity on $S^1 \vee S^2$ and constant map (basepoint of the wedge) on S^3.

For $q = 0, 1, 2$, $H_q(d) = d_*$ and $H^q(d) = d^*$ are group isomorphisms.

Let a' = the generator of $H^1(S^1 \vee S^2 \vee S^3)$
b' = the generator of $H^2(S^1 \vee S^2 \vee S^3)$
$a' = d^*(a'')$, $a'' \in H^1(S^1 \vee S^2)$
$b' = d^*(b'')$, $b'' \in H^2(S^1 \vee S^2)$.

Then $a'' \cup b'' \in H^3(S^1 \vee S^2) = 0$.

$\Rightarrow$ $a' \cup b' = d^*(a'') \cup d^*(b'') = d^*(a'' \cup b'') = d^*(0) = 0$.

$\Rightarrow$ $a' \cup b'$ does not generate $H^3(S^1 \vee S^2 \vee S^3)$

$\Rightarrow$ $H^*(S^1 \times S^2)$ and $H^*(S^1 \vee S^2 \vee S^3)$ are not isomorphic as rings.

$\Rightarrow$ $S^1 \times S^2$ and $S^1 \vee S^2 \vee S^3$ do not have the same homotopy type.

Remark:

Universal Coefficient Theorem for Cohomology
$\Rightarrow$ If X is a topological space of finite type, then the cohomology groups $H^*(X)$ determine the cohomology groups $H^*(X, G)$ for every abelian group G.

However, the cohomology ring $H^*(X)$ does not determine the cohomology ring $H^*(X, R)$:

P. J. Hilton and S. Wylie: Homology Theory, Example on p. 151: The cohomology rings $H^*(\mathbb{R}P^3)$ and $H^*(\mathbb{R}P^2 \vee S^3)$ are isomorphic, but the rings $H^*(\mathbb{R}P^3, \mathbb{Z}/2\mathbb{Z})$ and $H^*(\mathbb{R}P^2 \vee S^3, \underbrace{\ }_{\mathbb{Z}/2\mathbb{Z}})$ are not isomorphic.

Fibrations

Definition 16.3.

Let E and B be topological spaces, and let $p: E \to B$ be a continuous map. Let X be a topological space. We say that p has the <u>homotopy lifting property</u> with respect to X, if for every two continuous maps

$$\tilde{f}: X \to E \quad \text{and} \quad G: X \times I \to B$$

s.t. $p\tilde{f} = Gi$ (where $i: X \to X \times I$, $x \mapsto (x, 0)$),

there is a continuous map $\tilde{G}: X \times I \to E$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "G"'] \arrow[ur, dashed, "\tilde{G}"] & B \end{tikzcd}$$

Let $f : X \to B$, $x \mapsto G(x,0)$
and $g : X \to B$, $x \mapsto G(x,1)$.

Then $f = p\tilde{f}$, i.e., $\tilde{f}$ is a lifting of f,
and $g = p \circ \underbrace{\tilde{G}(\,,1)}_{\tilde{g}}$, i.e. $\tilde{g}$ is a lifting of g.

Now, $\tilde{G} : \tilde{f} \simeq \tilde{g}$ and $G : f \simeq g$.

<u>Definition 16.4.</u> Let $p : E \to B$ be a continuous map. If p has the homotopy lifting property with respect to every topological space X, then p is called a <u>fibration.</u> If $b \in B$, then $p^{-1}(b_0) = F$ is called the <u>fiber.</u>

<u>Example:</u> Let $E = B \times F$, and let

$$p : E = B \times F \to B, \quad (b,x) \mapsto b.$$

Then p is a fibration with fiber F:
Consider the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d, "i"'] & B \times F \arrow[d, "p"] \\ X \times I \arrow[r, "G"'] \arrow[ur, dashed, "?"] & B \end{tikzcd} \qquad i(x) = (x,0) \qquad (*)$$

Define
$$\tilde{G} : X \times I \to B \times F, \quad (x,t) \mapsto (G(x,t), q\tilde{f}(x)),$$

where $\quad q: B \times F \to F, \quad (b,x) \mapsto x.$

Then
$$\begin{aligned}\tilde{G}(i(x)) &= \tilde{G}(x,0)\\ &= (G(x,0), q\tilde{f}(x))\\ &= (p\tilde{f}(x), q\tilde{f}(x)) = \tilde{f}(x),\end{aligned}$$

and
$$\begin{aligned}p(\tilde{G}(x,t)) &= p(G(x,t), q\tilde{f}(x))\\ &= G(x,t).\end{aligned}$$

Thus $\tilde{G}$ makes each triangle in $(*)$ commute, and it follows that p is a fibration.

<u>Remark:</u> Often what we call a *fibration* is called a <u>*Hurewicz fibration*</u>. A continuous map $p: E \to B$ is called a <u>*Serre fibration*</u>, if it has the homotopy lifting property with respect to CW-complexes. Serre fibration is also called a <u>*weak fibration*</u>.

<u>**Theorem 16.5.**</u> Let $p: E \to B$ be a fibration, and let $b_0, b_1 \in B$. If B is path connected, then the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ have the same homotopy type.

<u>proof:</u> Rotman, Theorem 11.47.

<u>**Definition 16.6.**</u> A <u>*locally trivial bundle*</u> with fiber F is a map $p: E \to B$ for which there is an open cover $\mathcal{V}$ of B and homomorphisms
$$\varphi_V : V \times F \to p^{-1}(V)$$

for all $V \in \mathcal{V}$ s.t.

$$p\varphi_V(v,x) = v, \quad \text{for all } (v,x) \in V \times F.$$