Corollary 15.9. Let R be a commutative ring. Then $S^*(\ ,R)$ is a contravariant functor from Top to Graded Rings.

proof.
$$S^*(\ ,R) : X \mapsto S^*(X,R)$$

Let $f: X \to X'$. Then

$$f^\# : S^*(X',R) \to S^*(X,R).$$

Here $f^\#(S^n(X',R)) \subset S^n(X,R)$, and Lemma 15.8 $\Rightarrow f^\#$ is a ring homomorphism.

For $id_X : X \to X$, $id_X^\# = id : S^*(X,R) \to S^*(X,R)$.
For $f: X \to Y$, $g: Y \to Z$,

$$f^\# : S^*(Y,R) \to S^*(X,R), \quad g^\# : S^*(Z,R) \to S^*(Y,R),$$

and $(g \circ f)^\# = f^\# \circ g^\#$. $\square$

Lemma 15.10. Let $\varphi \in S^p(X,R)$, and let $\theta \in S^q(X,R)$. Then

$$\delta(\varphi \cup \theta) = \delta\varphi \cup \theta + (-1)^p \varphi \cup \delta\theta.$$

proof. The terms on each side have degree $d = p+q+1$. Let σ be a d-simplex. Then

$$(\sigma, \delta\varphi \cup \theta + (-1)^p \varphi \cup \delta\theta)$$

$$= (\sigma, \delta\varphi \cup \theta) + (-1)^p (\sigma, \varphi \cup \delta\theta)$$

$$= (\sigma\lambda_{p+1}, \delta\varphi)(\sigma\mu_q, \theta) + (-1)^p (\sigma\lambda_p, \varphi)(\sigma\mu_{q+1}, \delta\theta)$$

$$= (\partial(\sigma\lambda_{p+1}), \varphi)(\sigma\mu_q, \theta) + (-1)^p (\sigma\lambda_p, \varphi)(\partial(\sigma\mu_{q+1}), \theta) \qquad (*)$$

Recall that if $\tau : \Delta^n \to X$ is continuous, and if $n > 0$, then

$$\partial_n \tau = \sum_{i=0}^{n} (-1)^i \tau \varepsilon_i^n \in S_{n-1}(X).$$

Therefore, $(*)$ equals

$$\sum_{i=0}^{p+1} (-1)^i (\sigma \lambda_{p+1} \varepsilon_i, \varphi)(\sigma \mu_q, \theta) + \sum_{j=0}^{q+1} (-1)^{j+p} (\sigma \lambda_p, \varphi)(\sigma \mu_{q+1} \varepsilon_j, \theta). \quad (**)$$

Lemma 15.5 1 $\Rightarrow$ $\sigma \lambda_{p+1} \varepsilon_{p+1} = \sigma \lambda_{p+1} \lambda_p = \sigma \lambda_p$, and

$$\sigma \mu_{q+1} \varepsilon_0 = \sigma \mu_{q+1} \mu_q = \sigma \mu_q.$$

Thus: the term $p+1$ of the first sum equals the term 0 of the second sum, with opposite sign.

Thus $(**)$ equals

$$\sum_{i=0}^{p} (-1)^i (\sigma \lambda_{p+1} \varepsilon_i, \varphi)(\sigma \mu_q, \theta) + \sum_{j=1}^{q+1} (-1)^{j+p} (\sigma \lambda_p, \varphi)(\sigma \mu_{q+1} \varepsilon_j, \theta).$$

On the other hand,

$$(\sigma, \delta(\varphi \cup \theta)) = (\partial \sigma, \varphi \cup \theta)$$

$$= \sum_{i=0}^{d} (-1)^i (\sigma \varepsilon_i, \varphi \cup \theta)$$

$$= \sum_{i=0}^{d} (-1)^i (\sigma \varepsilon_i \lambda_p, \varphi)(\overset{(d-1)\text{-simplex}}{\sigma \varepsilon_i} \mu_q^{d-1}, \theta)$$

$$= \sum_{i=0}^{p} (-1)^i (\sigma \varepsilon_i \lambda_p, \varphi)(\sigma \varepsilon_i \mu_q, \theta) + \sum_{i=p+1}^{d} (-1)^i (\sigma \varepsilon_i \lambda_p, \varphi)(\sigma \varepsilon_i^d \mu_q^{d-1}, \theta)$$

$d-q=p+1$ Lemma 15.5.3

$$= \sum_{i=0}^{p} (-1)^i (\sigma \lambda_{p+1} \varepsilon_i, \varphi)(\sigma \mu_q, \theta) + \sum_{i=p+1}^{d} (-1)^i (\sigma \lambda_p, \varphi)(\sigma \mu_{q+1}^d \varepsilon^{q+1}_{\underbrace{i+q-d+1}_{i-p}}, \theta)$$

$j=-p+i$

$$= \sum_{i=0}^{p} (-1)^i (\sigma \lambda_{p+1} \varepsilon_i, \varphi)(\sigma \mu_q, \theta) + \sum_{j=1}^{q+1} (-1)^{j+p} (\sigma \lambda_p, \varphi)(\sigma \mu_{q+1} \varepsilon_j, \theta).$$

$\square$

Theorem 15.11. Let R be a commutative ring. Then

$$H^*(\ ;R) = \sum_{p\geq 0} H^p(\ ;R) : h\,Top \to \text{Graded Rings}$$

is a contravariant functor.

proof. Write

$$Z^*(X,R) = \sum_{p\geq 0} \overbrace{Z^p(X,R)}^{Z^p}$$

$$B^*(X,R) = \sum_{p\geq 0} \overbrace{B^p(X,R)}^{B^p}.$$

Let $\varphi \in Z^p$ and $\theta \in Z^q$. Then $\delta\varphi = 0 = \delta\theta$, and

$$\delta(\varphi \cup \theta) = \delta\varphi \cup \theta + (-1)^p \varphi \cup \delta\theta = 0.$$

$\Rightarrow$ $\varphi \cup \theta \in Z^{p+q}$. $\Rightarrow$ Z^* is a subring of $S^*(X,R)$.

Let then $\varphi \in Z^p$ and $\theta \in B^q$. Then $\delta\varphi = 0$, and $\theta = \delta\psi$, for some $\psi \in S^{q-1}(X,R)$. Then

$$\varphi \cup \theta = \varphi \cup \delta\psi = \pm\left(\delta(\varphi\cup\psi) - \underbrace{\delta\varphi \cup \psi}_{0}\right) = \pm\,\delta(\varphi\cup\psi).$$

$\Rightarrow$ $\varphi \cup \theta \in B^*$. Similarly, $\theta \cup \varphi \in B^*$.

$\Rightarrow$ B^* is a homogeneous ideal in Z^*.

Lemma 15.3 $\Rightarrow$ $H^*(X,R) = Z^*/B^*$ is a graded ring.

Here the multiplication in $H^*(X,R)$ is

$$[\varphi] \cup [\theta] = [\varphi \cup \theta],$$

where $[\ ]$ denotes cohomology class.

Let $f: X \to Y$ be a continuous map. Then f induces

$$f^\#: S^*(Y,R) \to S^*(X,R),$$

and $f^\#$ is a chain map and a ring homomorphism ($f^\#(\varphi \cup \theta) = f^\#\varphi \cup f^\#\theta$). But then $f^\#$ induces

$$f^*: H^*(Y,R) \to H^*(X,R),$$

and $f^*([\varphi]\cup[\theta]) = f^*([\varphi\cup\theta]) = [f^\#(\varphi\cup\theta)]$

$$= [f^\#\varphi \cup f^\#\theta] = [f^\#\varphi]\cup[f^\#\theta] = f^*[\varphi]\cup f^*[\theta].$$

Thus f^* is a ring homomorphism. Homotopy Axiom for Cohomology $\Rightarrow$ if $f,g: X \to Y$ are homotopic, then $f^* = g^*$.

Clearly, if $id_X: X \to X$ is the identity map of X, then $id_X^*: H^*(X,R) \to H^*(X,R)$ is the identity. Also, for continuous functions $f: X \to Y$ and $g: Y \to Z$, $(g\circ f)^* = f^* \circ g^*$. $\square$

The cup product

$$\cup: S^p(X,R) \times S^q(X,R) \to S^{p+q}(X,R), \quad (\varphi,\theta) \mapsto \varphi\cup\theta,$$

is a bilinear map. Then it induces a homomorphism

$$S^p(X,R) \otimes S^q(X,R) \to S^{p+q}(X,R)$$

also denoted by $\cup$ that makes the following diagram commute:

$$\begin{tikzcd}
S^p(X,R) \times S^q(X,R) \arrow[rr, "\nu"] \arrow[dr] & & S^p(X,R)\otimes S^q(X,R) \arrow[dl, "\cup"] \\
& S^{p+q}(X,R) &
\end{tikzcd}$$

ν = quotient map

Thus the cup product may be considered as a map

$$\cup : S^*(X,R) \otimes S^*(X,R) \to S^*(X,R).$$

Similarly, we get cup product

$$\cup : H^*(X,R) \otimes H^*(X,R) \to H^*(X,R).$$

Definition: Let X be a topological space, and let R be a commutative ring. The cohomology ring of X with coefficients R is

$$H^*(X,R) = \sum_{p \geq 0} H^p(X,R).$$

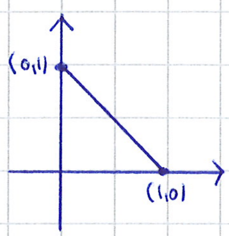
Example: Let X be a single point P. Let R be a commutative ring. Then

$$H^n(X,R) \cong (H^n(X) \otimes R) \oplus \overbrace{\operatorname{Tor}(\underbrace{H^{n+1}(X)}_{0 \text{ or } \mathbb{Z}}, R)}^{0}$$

$$\Rightarrow \begin{cases} H^0(X,R) \cong H^0(X) \otimes R = \mathbb{Z} \otimes R = R \\ H^n(X,R) \cong H^n(X) \otimes R = 0 \otimes R = 0 \end{cases}$$

$$\Rightarrow H^*(X,R) = \sum_{n \geq 0} H^n(X,R) = R$$

Example: The ring $S^*(X,R)$ does not have good commutativity properties. For example, let X be the 1-simplex Δ_1. Let $R = \mathbb{Z}$.



$S_0(\Delta_1)$ generated by points in Δ_1

Let $c \in \operatorname{Hom}(S_0(\Delta_1), \mathbb{Z}) = S^0(\Delta_1, \mathbb{Z})$,

$$(x,c) = c(x) = \begin{cases} 1, & \text{if } x = (1,0) \\ 0, & \text{if } x \in \Delta_1 - \{(1,0)\} \end{cases}$$

Extend c to $S_0(\Delta_1)$...

Let $d \in Hom(S_1(\Delta_1), \mathbb{Z}) = S^1(\Delta_1, \mathbb{Z})$,

$$(\sigma, d) = d(\sigma) = \begin{cases} 1, & \text{if } \sigma = id_{\Delta_1} \\ 0, & \text{if } \sigma \text{ is a 1-simplex}, \ \sigma \neq id_{\Delta_1} \end{cases}$$

Now, $\lambda_0 : \Delta_0 \to \Delta_1, \ 1 \mapsto (1,0)$
$\mu_0 : \Delta_0 \to \Delta_1, \ 1 \mapsto (0,1)$
$\lambda_1 : \Delta_1 \to \Delta_1, \ (t_0, t_1) \mapsto (t_0, t_1)$
$\mu_1 : \Delta_1 \to \Delta_1, \ (t_0, t_1) \mapsto (t_0, t_1)$,

and $c \cup d \in Hom(S_1(\Delta_1), \mathbb{Z}) = S^1(\Delta_1, \mathbb{Z})$.

$$(id_{\Delta_1}, c \cup d) = (id_{\Delta_1}\lambda_0, c)(id_{\Delta_1}\mu_1, d)$$
$$= (c(1,0))(d(id_{\Delta_1})) = 1 \cdot 1 = 1$$

and

$$(id_{\Delta_1}, d \cup c) = (id_{\Delta_1}\lambda_1, d)(id_{\Delta_1}\mu_0, c)$$
$$= (d(id_{\Delta_1}))(c(0,1)) = 1 \cdot 0 = 0$$

Augmentations:

Augmentation of $S_*(X)$: homom. $\varepsilon : S_0(X) \to \mathbb{Z}$
$\varepsilon(x) = 1 \quad \forall x \in X$

Augmentation of $S_*(X) \otimes S_*(X)$:

homom. $\varepsilon' : S_0(X) \otimes S_0(X) \to \mathbb{Z}$
$\varepsilon'(x \otimes y) = 1 \quad \forall \ x, y \in X$.

A chain map $\chi : S_*(X) \to S_*(X) \otimes S_*(X)$ preserves augmentation, if $\varepsilon' \chi_0 = \varepsilon$.

Definition: An augmentation preserving natural chain map $\chi : S_*(X) \to S_*(X) \otimes S_*(X)$ is called a diagonal approximation.

Eilenberg-Zilber Theorem $\Longrightarrow$ there is a natural chain equivalence

$$\zeta : S_*(X \times X) \longrightarrow S_*(X) \otimes S_*(X),$$

where

$$\zeta_0 : S_0(X \times X) \longrightarrow S_0(X) \otimes S_0(X)$$
$$(x, y) \longmapsto x \otimes y$$

Let $d : X \to X \times X$, $x \mapsto (x, x)$, be the diagonal map, then

$$d_\# : S_0(X) \longrightarrow S_0(X \times X).$$

Then the diagram

$$\begin{tikzcd} S_0(X) \arrow[r, "d_\#"] \arrow[drr, "\varepsilon"'] & S_0(X \times X) \arrow[r, "\zeta_0"] & S_0(X) \otimes S_0(X) \arrow[d, "\varepsilon'"] \\ & & \mathbb{Z} \end{tikzcd}$$

commutes:

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[drr, mapsto] & (x, x) \arrow[r, mapsto] & x \otimes x \arrow[d, mapsto] \\ & & 1 \end{tikzcd}$$

Thus $\zeta_0 d_\#$ is a diagonal approximation.

The following theorem says that, essentially, $\zeta_0 d_\#$ is the only diagonal approximation.

<u>Theorem 15.12.</u> Every two diagonal approximations are chain homotopic. Therefore, they induce the same homomorphisms in cohomology.

<u>Proof</u> By using acyclic models. We skip this. □

We next look for a formula for an Eilenberg-Zilber map

$$\zeta : S_*(X \times Y) \longrightarrow S_*(X) \otimes S_*(Y).$$

$$\zeta_0 : S_0(X \times Y) \longrightarrow S_0(X) \otimes S_0(Y), \quad (x,y) \mapsto x \otimes y.$$

$$\zeta_1 : S_1(X \times Y) \longrightarrow S_1(X) \otimes S_1(Y)$$

Let $\sigma : \Delta_1 \longrightarrow X \times Y$ be continuous,

let $\sigma(e_0) = (x_0, y_0)$, $\quad e_0 = (1,0)$

$\sigma(e_1) = (x_1, y_1)$, $\quad e_1 = (0,1)$.

The map ζ_1 must be such that the following diagram commutes:

$$\begin{tikzcd}
S_1(X \times Y) \arrow[r, "\partial_1"] \arrow[d, "\zeta_1"'] & S_0(X \times Y) \arrow[d, "\zeta_0"] \\
(S_1(X) \otimes S_0(Y)) \oplus (S_0(X) \otimes S_1(Y)) \arrow[r, "D_1"'] & S_0(X) \otimes S_0(Y)
\end{tikzcd}$$

Here D_1 is the differentiation on the tensor product, i.e.,

$$D_{i+j}(\alpha \otimes \beta) = \partial\alpha \otimes \beta + (-1)^i \alpha \otimes \partial\beta, \quad \text{where } \alpha \in S_i(X),\ \beta \in S_j(Y).$$

Let $\sigma \in S_1(X \times Y)$. Then

$$\zeta_0 \partial_1(\sigma) = \zeta_0(\sigma(e_1) - \sigma(e_0)) = \zeta_0((x_1, y_1) - (x_0, y_0))$$

$$= x_1 \otimes y_1 - x_0 \otimes y_0.$$

Let then $\pi' : X \times Y \to X$ and $\pi'' : X \times Y \to Y$

be projections. Let $\sigma' = \pi'\sigma$ and $\sigma'' = \pi''\sigma$.

Guess : $\xi_1(\sigma) = \sigma' \otimes y + x \otimes \sigma''$,

for some $x \in X$ and $y \in Y$.

All 0-chains are cycles $\Rightarrow$ $\partial y = 0 = \partial x$.
Then

$$\begin{aligned} D_1 \xi_1(\sigma) &= D_1(\sigma' \otimes y + x \otimes \sigma'') \\ &= \partial \sigma' \otimes y - \underbrace{\sigma' \otimes \partial y}_{0} + \underbrace{\partial x \otimes \sigma''}_{0} + x \otimes \partial \sigma'' \\ &= (\sigma'(e_1) - \sigma'(e_0)) \otimes y + x \otimes (\sigma''(e_1) - \sigma''(e_0)) \\ &= (\pi'\sigma(e_1) - \pi'\sigma(e_0)) \otimes y + x \otimes (\pi''\sigma(e_1) - \pi''\sigma(e_0)) \\ &= (x_1 - x_0) \otimes y + x \otimes (y_1 - y_0). \end{aligned}$$

Therefore, define $\xi_1(\sigma) = \sigma' \otimes y_1 + x_0 \otimes \sigma''$.

Then

$$\begin{aligned} D_1 \xi_1(\sigma) &= (x_1 - x_0) \otimes y_1 + x_0 \otimes (y_1 - y_0) \\ &= x_1 \otimes y_1 - x_0 \otimes y_1 + x_0 \otimes y_1 - x_0 \otimes y_0 \\ &= x_1 \otimes y_1 - x_0 \otimes y_0 \\ &= \xi_0 \partial_1(\sigma). \end{aligned}$$

Generally, the ξ_n are defined by the formula given by the following theorem:

Theorem 15.13. (Alexander – Whitney)

The map $\xi : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$ defined by

$$\xi_n(\sigma) = \sum_{i+j=n} \sigma' \lambda_i \otimes \sigma'' \mu_j ,$$

where

$$\sigma : \Delta_n \to X \times Y$$
$$\sigma' = \pi' \sigma \ , \quad \sigma'' = \pi'' \sigma$$
$\pi' : X \times Y \to X$, $\pi'' : X \times Y \to Y$ are the projections,

is a natural chain equivalence over $\xi_0 : (x, y) \mapsto x \otimes y$.

proof.

1. Naturality: Let $f : X \to X'$ and $g : Y \to Y'$ be continuous maps. Then the following diagram commutes:

$$\begin{tikzcd}
S_*(X \times Y) \arrow[r, "\xi"] \arrow[d, "(f\times g)_\#"'] & S_*(X) \otimes S_*(Y) \arrow[d, "f_\# \otimes g_\#"] \\
S_*(X' \times Y') \arrow[r, "\xi"'] & S_*(X') \otimes S_*(Y')
\end{tikzcd}$$

$$\begin{tikzcd}
\sigma \arrow[r, mapsto] \arrow[d, mapsto] & \sum\limits_{i+j=n} \sigma' \lambda_i \otimes \sigma'' \mu_j \arrow[d, mapsto] \\
(f \times g)_\# \sigma \arrow[dr, mapsto] & \sum\limits_{i+j=n} f_\# \sigma' \lambda_i \otimes g_\# \sigma'' \mu_j \arrow[d, phantom, "\parallel"] \\
& \sum\limits_{i+j=n} \underbrace{\pi'((f\times g)_\# \sigma)}_{f_\# \pi' \sigma = f_\# \sigma'} \lambda_i \otimes \underbrace{\pi''((f \times g)_\# \sigma)}_{g_\# \pi'' \sigma = g_\# \sigma''} \mu_j
\end{tikzcd}$$