Assume S_* and C_* are chain equivalent chain complexes. Then

1) $S_* \otimes G$ and $C_* \otimes G$ are chain equivalent

2) $\mathrm{Hom}(S_*, G)$ and $\mathrm{Hom}(C_*, G)$ are chain equivalent,

where G is any abelian group.

<u>Theorem 14.8.</u> (Universal Coefficients Theorem for Cohomology)

Let X be a topological space of finite type, and let G be an abelian group. Then, for every $n \geq 0$, there is an exact sequence

$$0 \to H^n(X) \otimes G \xrightarrow{\alpha} H^n(X; G) \to \mathrm{Tor}(H^{n+1}(X), G) \to 0,$$

where $\alpha : [z] \otimes g \mapsto [zg]$

and $zg : \sigma \mapsto \underbrace{z(\sigma)}_{\in \mathbb{Z}} g$, for a singular n-simplex σ.

The sequence splits:

$$H^n(X; G) \cong (H^n(X) \otimes G) \oplus \mathrm{Tor}(H^{n+1}(X), G).$$

<u>proof.</u> Since X has finite type, Lemma 14.7 $\Rightarrow$ there is a free chain complex C_* of finite type with $H_*(C_*) = H_*(X)$.

Let $A^* = \mathrm{Hom}(C_*, \mathbb{Z})$. Then A^* is a free chain complex: Let $m = \mathrm{rank}\, C_n$. Let $\{x_1, \ldots, x_m\}$ be a basis for C_n. Define

$$F : \mathrm{Hom}(C_n, \mathbb{Z}) \to \mathbb{Z}^m, \quad f \mapsto (z_1, \ldots, z_m),$$

if $f(x_i) = z_i \quad \forall\, 1 \leq i \leq m$.

Clearly, F is injective and surjective. Let $f, g \in \mathrm{Hom}(C_n, \mathbb{Z})$, $f(x_i) = z_i$, $g(x_i) = w_i$, $1 \le i \le m$. Then $(f+g)(x_i) = z_i + w_i$ $\forall\, 1 \le i \le m$. Thus

$$F(f+g) = (z_1+w_1, \ldots, z_m+w_m) = F(f) + F(g).$$

It follows that F is a homomorphism. Hence F is an isomorphism, and $A^n \cong \mathbb{Z}^m$.

The universal coefficients theorem for homology now applies to A^*. Thus there is an exact split sequence

$$0 \to H^n(A^*) \otimes G \xrightarrow{\alpha} H^n(A^* \otimes G) \to \mathrm{Tor}(H^{n+1}(A^*), G) \to 0.$$

$$\Big[\ 0 \to A^0 \xrightarrow{\delta^0} A^1 \xrightarrow{\delta^1} A^2 \to \cdots \to A^{n-1} \xrightarrow{\delta^{n-1}} A^n \xrightarrow{\delta^n} A^{n+1} \to \cdots$$

$$H^n(A^*) = \ker \delta^n / \mathrm{im}\, \delta^{n-1}\ \Big]$$

Here

$$\begin{aligned} H^n(A^*) &= H^n(\mathrm{Hom}(C_*, \mathbb{Z})) \\ &= H^n(\mathrm{Hom}(S_*(X), \mathbb{Z})), \quad S_*(X) \text{ chain equiv. to } C_* \\ &= H^n(X). \end{aligned}$$

Lemma 14.7 $\Rightarrow$

$$A^* \otimes G = \mathrm{Hom}(C_*, \mathbb{Z}) \otimes G \cong \mathrm{Hom}(C_*, G) \cong \mathrm{Hom}(S_*(X), G).$$

Thus

$$H^n(A^* \otimes G) \cong H^n(\mathrm{Hom}(S_*(X), G)) = H^n(X; G).$$

Thus there is a split exact sequence

$$0 \to H^n(X) \otimes G \to H^n(X; G) \to \mathrm{Tor}(H^{n+1}(X), G) \to 0.$$

$\square$

Recall: $S_i(X) \otimes S_j(Y)$ is a free abelian group with basis all $\sigma_i \otimes \tau_j$, where σ_i is an i-simplex in X and τ_j is an j-simplex in Y.

Definition 14.9. Let R be a commutative ring. Let $\varphi \in \mathrm{Hom}(S_m(X), R)$ and $\theta \in \mathrm{Hom}(S_n(Y), R)$.
Let

$$\varphi \otimes \theta : S_*(X) \otimes S_*(Y) \to R,$$

$$\sigma_i \otimes \tau_j \mapsto \begin{cases} \varphi(\sigma_m)\,\theta(\tau_n), & \text{if } i=m,\ j=n \\ 0, & \text{otherwise} \end{cases}$$

Then $\varphi \otimes \theta \in \mathrm{Hom}(S_*(X) \otimes S_*(Y), R)$.

Theorem 14.10. (Künneth Formula for Cohomology)

Let X and Y be topological spaces of finite type. Then there is a short exact sequence that splits:

$$0 \to \sum_{i+j=n} H^i(X) \otimes H^j(Y) \xrightarrow{\alpha'} H^n(X \times Y) \to \sum_{p+q=n+1} \mathrm{Tor}(H^p(X), H^q(Y)) \to 0,$$

where $\alpha' : [\varphi_i] \otimes [\theta_j] \mapsto [\xi^{\#}(\varphi_i \otimes \theta_j)]$ and $\xi : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$ is the Eilenberg-Zilber chain equivalence.

proof. Since X and Y have finite type, it follows from Lemma 14.6, that there are free chain complexes C_* and E_* of finite type chain equivalent to $S_*(X)$ and $S_*(Y)$, respectively. Consider the diagram

$$\begin{array}{ccc} H^i(X) \otimes H^j(Y) & \xrightarrow{\alpha'} & H^{i+j}(X \times Y) \\ \cong \downarrow & & \downarrow \cong \\ H^i(C_*) \otimes H^j(E_*) & \xrightarrow[\alpha']{} & H^{i+j}(\mathrm{Hom}(C_* \otimes E_*, \mathbb{Z})) \end{array} \qquad (*)$$

C_*, E_* chain eq. to $S_*(X), S_*(Y)$, respectively
$\Rightarrow$ $C_* \otimes E_*$ chain eq. to $S_*(X) \otimes S_*(Y)$.

The vertical maps are isomorphisms induced by chain equivalences. Let the chain equivalences be $f_* : C_* \to S_*(X)$ and $g_* : E_* \to S_*(Y)$. Then

$$\begin{tikzcd}
{[\varphi_i] \otimes [\theta_j]} \arrow[r, mapsto] \arrow[d, mapsto] & {[\xi^{\#}(\varphi_i \otimes \theta_j)]} \arrow[d, mapsto, "\leftarrow \text{check this}"] \\
{[\varphi_i f] \otimes [\theta_j g]} \arrow[r, mapsto] & {[(\varphi_i f) \otimes (\theta_j g)]}
\end{tikzcd}$$

and the diagram $(*)$ commutes.

Since C_* is a free chain complex of finite type, it follows that $\forall n : C_n \cong \mathbb{Z}^{k(n)}$ for some $k(n) \in \mathbb{N} \cup \{0\}$. Then

$$\mathrm{Hom}(C_n, \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}^{k(n)}, \mathbb{Z}) \overset{\text{HW 10}}{\cong} \big(\underbrace{\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})}_{\cong \mathbb{Z}}\big)^{k(n)} \cong (\mathbb{Z})^{k(n)}.$$

Thus $\mathrm{Hom}(C_*, \mathbb{Z})$ is a free chain complex. Similarly, $\mathrm{Hom}(E_*, \mathbb{Z})$ is a free chain complex. Therefore, Künneth Theorem applies to the bottom sequence, i.e., there is a split short exact sequence

$$0 \to \sum_{i+j=n} H^i(C_*) \otimes H^j(E_*) \xrightarrow{\alpha'} H^n(\mathrm{Hom}(C_* \otimes E_*, \mathbb{Z}))$$
$$\to \sum_{p+q=n+1} \mathrm{Tor}(H^p(C_*), H^q(E_*)) \to 0.$$

The claim follows, since the diagram $(*)$ commutes.

$\square$

A-modules

Definition 14.11. Let A be a commutative ring with unity. Let M be an abelian group equipped with scalar multiplication

$$A \times M \to M, \quad (a, m) \mapsto am,$$

such that the following identities hold for all $m, m' \in M$ and for all $a, a' \in A$:

1) $a(m + m') = am + am'$

2) $(a + a')m = am + a'm$

3) $(aa')m = a(a'm)$

4) $1m = m$, where $1 \in A$ is the unity.

Then M is called an A-module.

Then:

1) If $A = \mathbb{Z}$, an A-module is an abelian group, no more structure comes from conditions 1 - 4.

2) Assume A is a field. Then, by definition, an A-module is a vector space over A.

3) The ring A is an A-module when the multiplication on A is considered as scalar multiplication.

<u>Example:</u> Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function. The support of f is the closure of the set $\{x \in \mathbb{R} \mid f(x) \neq 0\}$. Let A be the set of all continuous functions $\mathbb{R} \to \mathbb{R}$ with compact support. Adding and multiplying functions makes A a ring. This ring has no unity, since the function $c_1: \mathbb{R} \to \mathbb{R}$ has non-compact support, and hence is not in A. (c_1 = the constant function $x \mapsto 1 \ \forall x$)

Let $M_1, \ldots, M_n$ be A-modules. Consider them as abelian groups, and form the direct sum

$$M_1 \oplus \cdots \oplus M_n.$$

Defining

$$a(m_1, \ldots, m_n) = (am_1, \ldots, am_n)$$

makes $M_1 \oplus \cdots \oplus M_n$ an A-module called the <u>direct sum</u> of $M_1, \ldots, M_n$. The direct sum $A^{(n)}$ of n copies of A is called a <u>free A-module</u>. Let $e_i \in A^{(n)}$ be the n-tuple having 1 in the ith position and zeros elsewhere. Then every $m \in A^{(n)}$ can be uniquely written as

$$m = \sum a_i e_i, \quad \text{where } a_i \in A \ \forall i.$$

<u>Definition 14.12.</u> Let $\{b_1, \ldots, b_n\} \in A^{(n)}$. If $\forall m \in A^{(n)}$ has a unique expression of the form $m = \sum a_i b_i$, where $a_i \in A \ \forall i$, then $\{b_1, \ldots, b_n\}$ is called an <u>A-basis</u> of $A^{(n)}$.

Thus $\{e_1, \ldots, e_n\}$ is an example of an A-basis of $A^{(n)}$.

<u>Definition 14.13.</u> Let M be an A-module, and let $p \geq 0$. The <u>p^{th} exterior power</u> $\Lambda^p M$ of M is the abelian group with the following presentation:

<u>Generators:</u> $A \times \underbrace{M \times \cdots M}_{p \text{ times}}$

<u>Relations:</u> For all $a, a' \in A$ and for all $m_i, m_i' \in M$,

$$(a, m_1, \ldots, m_i + m_i', \ldots, m_p) = (a, m_1, \ldots, m_i, \ldots, m_p) + (a, m_1, \ldots, m_i', \ldots, m_p) \quad \forall i,$$

$$(a + a', m_1, \ldots, m_p) = (a, m_1, \ldots, m_p) + (a', m_1, \ldots, m_p)$$

$$(aa', m_1, \ldots, m_i, \ldots, m_p) = (a, m_1, \ldots, a'm_i, \ldots, m_p) \quad \forall i,$$

$$(a, m_1, \ldots, m_p) = 0 \quad \text{if} \quad m_i = m_j \text{ for some } i \neq j.$$

Here:

$p = 0$: $\Lambda^0 M = A$

$p = 1$: <u>Exercise:</u> Show that $\Lambda^1 M$ is isomorphic to M.

Let F be the free abelian group with basis $A \times \underbrace{M \times \cdots \times M}_{p}$, let S be the subgroup generated by the relations. Denote

$$a m_1 \wedge \cdots \wedge m_p = (a, m_1, \ldots, m_p) + S.$$

Then every element of $\Lambda^p M$ has a (not necessarily unique) expression of the form

$$\sum_j a_j m_1^j \wedge \cdots \wedge m_p^j, \quad \text{where } a_j \in A, \ m_i^j \in M.$$

The multiplication

$$A \times \Lambda^p M \to \Lambda^p M, \quad (a, \sum_j a_j m_1^j \wedge \cdots \wedge m_p^j) \mapsto \sum_j (a a_j) m_1^j \wedge \cdots \wedge m_p^j$$

satisfies conditions 1-4 of Definition 14.11. Thus $\Lambda^p M$ is an A-module. It follows from the last relation, that

$$m \wedge m = (1, m, m) + S = 0,$$

for any $m \in M$. Thus, if $m, m' \in M$, then

$$\begin{aligned} 0 &= (m+m') \wedge (m+m') \\ &= (1, m+m', m+m') + S \\ &= (1, m, m+m') + (1, m', m+m') + S \\ &= (1, m, m) + (1, m, m') + (1, m', m) + (1, m', m') + S \\ &= \underbrace{m \wedge m}_{0} + m \wedge m' + m' \wedge m + \underbrace{m' \wedge m'}_{0} \\ &= m \wedge m' + m' \wedge m. \end{aligned}$$

Thus $\quad m \wedge m' = - m' \wedge m, \quad \forall m, m' \in M.$

Similarly: Let $p \geq 2$. Interchange two factors of $m_1 \wedge \cdots \wedge m_p$ $\Rightarrow$ the sign gets reversed.

Let $M = A^{(n)}$ be a free A-module with an A-basis $\{e_1, \ldots, e_n\}$. Then it can be shown that $\Lambda^p M$ is also a free A-module with an A-basis

$$\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 1 \le i_1 < i_2 < \cdots < i_p \le n\}.$$

Then every element of $\Lambda^p(A^{(n)})$ has a unique expression

$$\sum a_{i_1, \ldots, i_p} e_{i_1} \wedge \cdots \wedge e_{i_p},$$

where $a_{i_1, \ldots, i_p} \in A$ and $1 \le i_1 < i_2 < \cdots < i_p \le n$. If $p > n$, then $\Lambda^p M = 0$. A basis of $\Lambda^p(A^{(n)})$ has $\binom{n}{p}$ elements.

15. Cohomology Rings

<u>Definition 15.1.</u> Let R be a ring, and let R^n, $n \geq 0$, be additive subgroups of R satisfying the following conditions:

1) $R = \sum_{n \geq 0} R^n$ (direct sum of additive groups)

2) for all $n, m \geq 0$, $R^n R^m \subset R^{n+m}$
(i.e., if $x \in R^n$ and $y \in R^m$, then $xy \in R^{n+m}$).

Then R is called a <u>graded ring.</u>

<u>Example:</u> Let A be a commutative ring. The <u>polynomial ring</u> $R = A[x]$ consists of polynomials $\sum_{m=0}^{\infty} p_m x^m$, where $p_m \in A$ $\forall m$ and $p_m \neq 0$ for only finitely many m. Adding and multiplying polynomials makes $A[X]$ a commutative ring. Setting $R^n = \{a x^n \mid a \in A\}$ makes R a graded ring.

<u>Example:</u> Let $R = A[x_1, \ldots, x_p]$ be a polynomial ring in several variables. Setting $R^n = \{\sum a x_1^{e_1} \cdots x_p^{e_p} \mid a \in A, \sum e_i = n\}$ makes R a graded ring. Then R^n is generated by monomials whose total degree equals n.

<u>Example:</u> Define

$$\Lambda^p M \times \Lambda^q M \to \Lambda^{p+q} M, \; ((m_1 \wedge \ldots \wedge m_p), (m'_1 \wedge \ldots \wedge m'_q)) \mapsto m_1 \wedge \ldots \wedge m_p \wedge m'_1 \wedge \ldots \wedge m'_q,$$

where M is an A-module. Then $\sum_{p \geq 0} \Lambda^p M$ is a graded ring.

<u>Definition 15.2.</u> Let $R = \Sigma R^n$ be a graded ring. If $x \in R^n$, we say that x has <u>degree n</u>. Such elements are called <u>homogeneous</u>. An ideal I or a subring S of R is called homogeneous, if it is generated by homogeneous elements.

<u>Recall the following:</u> Let R be a ring and let S be a nonempty subset of R that is closed under the operations of addition and multiplication in R. If S is a ring under these operations, then S is called a <u>subring</u> of R. A subring I of a ring R is a <u>left ideal</u> provided

(assuming ring does not necessarily have an identity, $\{0\}$ is an ideal)

$$r \in R, \; x \in I \Rightarrow rx \in I,$$

I is a <u>right ideal</u> provided

$$r \in R, \; x \in I \Rightarrow xr \in I.$$

I is an <u>ideal</u> if it is both a left and right ideal.

<u>Exercise:</u> Let $R = \Sigma R^n$ be a graded ring, and let I be an ideal of R. Show that the following are equivalent:

1) I is homogeneous

2) $I = \Sigma(I \cap R^n)$.

<u>Notice:</u> The definition of degree in a graded ring is not the same as the usual definition of a degree in a polynomial ring. Usually, the degree of a polynomial $a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ is n, if $a_n \neq 0$. However, the degree in the sense of graded rings is only defined for monomials. For graded ring, the zero element has degree n, for all $n \geq 0$.

The degree of $1 \in R = \sum R^n$:

Assume $1 = e_0 + \dots + e_k$, where $e_i \in R^i$. Assume $a_n \in R^n$. Then

$$a_n = a_n \cdot 1 = 1 \cdot a_n = e_0 a_n + \dots + e_k a_n \in R^n \cap (R^n \oplus \dots \oplus R^{n+k}) = R^n.$$

Thus $e_i a_n = 0 \;\; \forall i \geq 1$, and $e_0 a_n = a_n$.
Thus $a = e_0 a \;\; \forall a \in R$.
Similarly, $a = a e_0 \;\; \forall a \in R$.

$\Rightarrow$ e_0 is a two-sided identity.
In a ring two-sided identities are unique.
Therefore, $e_0 = 1$. Thus $1 = e^0 \in R^0$ and the degree of 1 equals zero.

Lemma 15.3. Let R be a graded ring, and let I be a homogeneous ideal in $R \; (= \sum R^n)$.
Then

$$R/I = \sum (R^n + I)/I$$

is a graded ring.

proof. I homogeneous $\Rightarrow I = \sum (I \cap R^n)$. As abelian groups,

$$R/I = \sum R^n / \sum (I \cap R^n) \cong \sum (R^n / I \cap R^n) \cong \sum (R^n + I)/I.$$

Also,

$$(R^n + I)/I \cdot (R^m + I)/I$$

$$= (R^n R^m + \underbrace{R^n I + I R^m + I \cdot I}_{\subset I, \text{ since } I \text{ is an ideal}})/I \subset (R^n R^m + I)/I$$

$$\subset (R^{n+m} + I)/I.$$

$\square$

Every commutative ring is an abelian group under addition. Thus $H^n(X;R)$ makes sense when R is a commutative ring. We will show that $H^*(X;R) = \sum H^n(X;R)$ has a multiplication called <u>cup product</u> that makes it a graded ring.

<u>Definition 15.4.</u> Let $0 \le i \le d$. Define affine maps $\lambda_i, \mu_i : \Delta^i \to \Delta^d$ by

$$\lambda_i : (t_0, \dots, t_i) \mapsto (t_0, \dots, t_i, 0, \dots, 0)$$

and

$$\mu_i : (t_0, \dots, t_i) \longmapsto (0, \dots, 0, t_0, \dots, t_i).$$

The maps λ_i and μ_i are called the <u>front face</u> and the <u>back face</u>, respectively.

Often one uses the notation

$$\lambda_i^d = \lambda_i \quad \text{and} \quad \mu_i^d = \mu_i .$$

Then $\lambda_d^d = \mu_d^d = id : \Delta^d \to \Delta^d$,

$$\lambda_0^d(1) = (1, 0, \dots, 0) = e_0$$

$$\mu_0^d(1) = (0, \dots, 0, 1) = e_d .$$

Recall the face maps $\varepsilon_i^{d+1} : \Delta^d \to \Delta^{d+1}$, $0 \le i \le d+1$:

$$\varepsilon_0(t_0, \dots, t_d) = (0, t_0, \dots, t_d) = \mu_d^{d+1}(t_0, \dots, t_d).$$

If $1 \le i \le d+1$, then

$$\varepsilon_i(t_0, \dots, t_d) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_d).$$

Lemma 15.5.

1) Let $\varepsilon_i^{d+1} : \Delta^d \to \Delta^{d+1}$ be the i^{th} face map. Then

$$\mu_d^{d+1} = \varepsilon_0^{d+1} \quad \text{and} \quad \lambda_d^{d+1} = \varepsilon_{d+1}^{d+1}.$$

2) $\mu_{m+k}^{d} \mu_k^{m+k} = \mu_k^d$, $\quad \lambda_{n+m}^{d} \lambda_n^{n+m} = \lambda_n^d$, $\quad \mu_{m+k}^{n+m+k} \lambda_m^{m+k} = \lambda_{n+m}^{n+m+k} \mu_m^{n+m}$.

3) $$\varepsilon_i^{d+1} \lambda_p^d = \begin{cases} \lambda_{p+1}^{d+1} \varepsilon_i^{p+1}, & \text{if } i \leq p \\ \lambda_p^{d+1}, & \text{if } i \geq p+1. \end{cases}$$

$$\varepsilon_i^{d+1} \mu_q^d = \begin{cases} \mu_q^{d+1}, & \text{if } i \leq d-q \\ \mu_{q+1}^{d+1} \varepsilon_{i+q-d}^{q+1}, & \text{if } i \geq d-q+1. \end{cases}$$

proof. HW $\square$

Notation: Let X be a topological space, and let G be an abelian group. We have the following notation:

$$S^n(X,G) = \mathrm{Hom}(S_n(X), G)$$

$$S^*(X,G) = \sum_{n \geq 0} S^n(X,G)$$

Let $\varphi \in S^n(X,G)$, and let $c \in S_n(X)$. We write

$$(c, \varphi) = \varphi(c) \in G.$$

Let $\delta = \delta^n = \partial_{n+1}^{\#} : \mathrm{Hom}(S_n(X), G) \to \mathrm{Hom}(S_{n+1}(X), G)$

$$\varphi \longmapsto \varphi\, \partial_{n+1}$$

If $c' \in S_{n+1}(X)$, then

$$\begin{aligned} (c', \delta(\varphi)) &= (c', \varphi\,\partial_{n+1}) = (\varphi\,\partial_{n+1})(c') \\ &= \varphi(\partial_{n+1} c') = (\partial_{n+1} c', \varphi). \end{aligned}$$

Let $f : X \to Y$ be a continuous function, and let $\varphi \in S^n(Y, G)$. Then

$$f^\# : S^n(Y, G) \to S^n(X, G), \quad h \mapsto h f_\#,$$

and

$$(c, f^\#(\varphi)) = (c, \varphi f_\#) = (\varphi f_\#)(c)$$
$$= \varphi(f_\#(c)) = (f_\#(c), \varphi),$$

where $f_\# : S_n(X) \to S_n(Y)$ is induced by f as usual.

If c is an n-simplex σ, then $f_\#(c) = f_\#(\sigma) = f\sigma$, and

$$(\sigma, f^\#(\varphi)) = (f\sigma, \varphi).$$

The n-simplexes $\sigma : \Delta^n \to X$ generate $S_n(X)$. Then every $\varphi \in S^n(X, G)$ is determined by the $(\sigma, \varphi) = \varphi(\sigma)$, where σ is an n-simplex.

<u>Definition 15.6.</u> Let X be a topological space, and let R be a commutative ring. Let $\varphi \in S^n(X, R)$, and let $\theta \in S^m(X, R)$. The <u>cup product</u> $\varphi \cup \theta \in S^{n+m}(X, R)$ of φ and θ is defined by

$$(\sigma, \varphi \cup \theta) = (\sigma \lambda_n, \varphi)(\sigma \rho_m, \theta),$$

for any $(n+m)$-simplex $\sigma \in S_{n+m}(X)$. The right side is a product of two elements in R.

The cup product defines a function

$$S^*(X, R) \times S^*(X, R) \to S^*(X, R),$$

$$\left(\sum \varphi_i\right) \cup \left(\sum \theta_j\right) = \sum_{i,j} \varphi_i \cup \theta_j,$$

where $\varphi_i \in S^i(X, R)$ and $\theta_j \in S^j(X, R)$.

Lemma 15.7. Let X be a topological space, and let R be a commutative ring. Then $S^*(X,R) = \sum S^n(X,R)$ is a graded ring under cup product.

proof. Every $S^n(X,R)$ is an abelian group, hence also $S^*(X,R)$ is an abelian group. To show that $S^*(X,R)$ is a ring we have to check that the multiplication on $S^*(X,R)$ is associative and that the multiplication is distributive with respect to addition. We also show that there is an element e in $S^0(X,R)$ that is the multiplicative identity.

1. Left distributivity:

Let $\varphi \in S^n(X,R)$ and $\theta, \psi \in S^m(X,R)$. We show that

$$\varphi \cup (\theta + \psi) = (\varphi \cup \theta) + (\varphi \cup \psi).$$

Let σ be an $(n+m)$-simplex. Then

$$\begin{aligned}(\sigma, \varphi \cup (\theta + \psi)) &= (\sigma \lambda n, \varphi)(\sigma \rho m, \theta + \psi) \\ &= (\sigma \lambda n, \varphi)((\sigma \rho m, \theta) + (\sigma \rho m, \psi)) \\ &= (\sigma \lambda n, \varphi)(\sigma \rho m, \theta) + (\sigma \lambda n, \varphi)(\sigma \rho m, \psi) \\ &= (\sigma, \varphi \cup \theta) + (\sigma, \varphi \cup \psi).\end{aligned}$$

$$\therefore \varphi \cup (\theta + \psi) = (\varphi \cup \theta) + (\varphi \cup \psi)$$

Right distributivity can be proved similarly.

2. Associativity:

Let $\varphi \in S^n(X,R)$, $\theta \in S^m(X,R)$ and $\psi \in S^k(X,R)$. We show that

$$\varphi \cup (\theta \cup \psi) = (\varphi \cup \theta) \cup \psi.$$

Let σ be an $(n+m+k)$-simplex. Then

$$
\begin{aligned}
(\sigma, \varphi \cup (\theta \cup \psi)) &= (\sigma \lambda_n, \varphi)(\sigma \mu_{m+k}, \theta \cup \psi) \\
&= (\sigma \lambda_n, \varphi)(\sigma \mu_{m+k} \lambda_m, \theta)(\sigma \mu_{m+k} \mu_k, \psi)
\end{aligned}
$$

and

$$
\begin{aligned}
(\sigma, (\varphi \cup \theta) \cup \psi) &= (\sigma \lambda_{n+m}, \varphi \cup \theta)(\sigma \mu_k, \psi) \\
&= (\sigma \lambda_{n+m} \lambda_n, \varphi)(\sigma \lambda_{n+m} \mu_m, \theta)(\sigma \mu_k, \psi).
\end{aligned}
$$

Lemma 15.5 2) $\Rightarrow$ $\lambda_{n+m} \lambda_n = \lambda_n$, $\mu^{n+m+k}_{m+k} \lambda^{m+k}_m = \lambda^{n+m+k}_{n+m} \mu^{n+m}_m$, $\mu_{m+k} \mu_k = \mu_k$.

$$\therefore \ \varphi \cup (\theta \cup \psi) = (\varphi \cup \theta) \cup \psi.$$

3. <u>Multiplicative identity:</u>

Define $e \in S^0(X, R) = \mathrm{Hom}(S_0(X), R)$ by

$$(x, e) = 1, \quad \text{for all } x \in X,$$

where the points of X are identified with 0-simplices in X. Let $\varphi \in S^n(X, R)$, and let σ be an n-simplex. Then

$$(\sigma, e \cup \varphi) = \underbrace{(\sigma \lambda_0, e)}_{1}(\underbrace{\sigma \mu_n}_{\sigma}, \varphi) = (\sigma, \varphi).$$

Thus $e \cup \varphi = \varphi$. Similarly, $\varphi \cup e = \varphi$. Thus e is the multiplicative identity for $S^*(X, R)$.

$\square$

<u>Lemma 15.8.</u> Let $f: X \to X'$ be a continuous map. Then

$$f^\#(\varphi \cup \theta) = f^\#(\varphi) \cup f^\#(\theta).$$

Let $e \in S^0(X, R)$ and $e' \in S^0(X', R)$ be the multiplicative identities of $S^*(X, R)$ and $S^*(X', R)$, respectively. Then

$$f^\#(e') = e.$$

proof. Let $\varphi \in S^p(X', R) = \mathrm{Hom}(S_p(X'), R)$, and let $\theta \in S^q(X', R) = \mathrm{Hom}(S_q(X'), R)$. Let σ be a $(p+q)$-simplex in X. Then

$$
\begin{aligned}
(\sigma, f^{\#}(\varphi \cup \theta)) &= (f\sigma, \varphi \cup \theta) \\
&= (f\sigma \lambda p, \varphi)(f\sigma \mu q, \theta) \\
&= (\sigma \lambda p, f^{\#}\varphi)(\sigma \mu q, f^{\#}\theta) \\
&= (\sigma, f^{\#}\varphi \cup f^{\#}\theta).
\end{aligned}
$$

Let $x \in X$. Then $(x, f^{\#} e') = (f(x), e') = 1$. Since this holds for every $x \in X$, it follows that $f^{\#} e' = e$.

$\square$

The category of graded rings

- Objects: graded rings
- Morphisms: Let $R = \sum_{n \geq 0} R^n$ and $S = \sum_{n \geq 0} S^n$ be graded rings. Morphisms $R \to S$ are ring homomorphisms $f: R \to S$ s.t. $f(R^n) \subset S^n$ $\forall n$.

($f: R \to S$ is a ring homomorphism, if

1) $f(a+b) = f(a) + f(b)$ $\forall a, b \in R$

2) $f(ab) = f(a) f(b)$ $\forall a, b \in R$

3) $f(1_R) = 1_S$, where 1_R and 1_S are the mult. identities of R and S, respectively)

Denote this category by Graded Rings.