Get rid of the negative indices: Define

$$A^n = A_{-n}, \quad \delta^n = d_{-n} = \partial^{\#}_{n+1}$$

Then $\quad A^n = \mathrm{Hom}(S_n(X), G)$

$$H^n(S_*(X), G) = H_{-n}(A_*) = \ker \partial^{\#}_{n+1} / \operatorname{im} \partial^{\#}_n .$$

$$\delta^n = \partial^{\#}_{n+1}, \quad \delta^n \phi = \phi \partial_{n+1} .$$

Thus we have the sequence

$$0 \to A^0 \xrightarrow{\delta^0} A^1 \xrightarrow{\delta^1} A^2 \to \cdots \to A^{n-1} \xrightarrow{\delta^{n-1}} A^n \xrightarrow{\delta^n} A^{n+1} \to \cdots$$

Definition 13.2. Let X be a topological space and let G be an abelian group. Let $n \geq 0$. Then

$\mathrm{Hom}(S_n(X), G)$ = the group of singular n-cochains in X with coefficients in G

$Z^n(X; G) = \ker \delta^n$ = the group of n-cocycles

$B^n(X; G) = \operatorname{im} \delta^{n-1}$ = the group of n-coboundaries

$H^n(X; G) = Z^n(X;G)/B^n(X;G) = \ker \delta^n / \operatorname{im} \delta^{n-1} = \ker \partial^{\#}_{n+1} / \operatorname{im} \partial^{\#}_n$

= the n^{th} cohomology group of X with coefficients in G.

If $\xi \in Z^n(X;G)$, then the coset $\xi + B^n(X;G)$ is an element in $H^n(X;G)$ called a cohomology class and denoted by $[\xi]$.

Theorem 13.3. Let G be an abelian group, and let $n \geq 0$. Then

$$H^n(\ ;G) : \mathrm{Top} \to \mathrm{Ab}$$

is a contravariant functor.

proof. By definition, $H^n(X;G) = Z^n(X;G)/B^n(X;G)$ is an abelian group. Let $f : X \to Y$ be a continuous function. Then it induces the chain map $f_\# : S_*(X) \to S_*(Y)$. Thus the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & S_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & S_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & S_n(Y) \arrow[r, "\partial'"'] & S_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

Applying $\mathrm{Hom}(\ ,G)$ to the diagram gives the following commutative diagram:

$$\begin{tikzcd}
\cdots & \mathrm{Hom}(S_n(X),G) \arrow[l] & \mathrm{Hom}(S_{n-1}(X),G) \arrow[l, "\delta"'] & \cdots \arrow[l] \\
\cdots & \mathrm{Hom}(S_n(Y),G) \arrow[l] \arrow[u, "f^\#"] & \mathrm{Hom}(S_{n-1}(Y),G) \arrow[l, "\delta'"] \arrow[u, "f^\#"] & \cdots \arrow[l]
\end{tikzcd}$$

where $f^\# : h \mapsto hf$ for every $h : S_n(Y) \to G$.

Check: $f^\#(Z^n(Y;G)) \subset Z^n(X;G)$

$$f^\#(B^n(Y;G)) \subset B^n(X;G)$$

Then $f^\#$ induces a homomorphism

$$f^* : H^n(Y;G) \to H^n(X;G),$$
$$\xi + B^n(Y;G) \mapsto f^\#(\xi) + B^n(X;G) = \xi f + B^n(X;G).$$

It remains to check that $(fg)^* = g^* f^*$ and $(id)^* = id$. This is left as an exercise. □

We next check that the Eilenberg-Steenrod axioms hold for cohomology.

<u>Theorem 13.4.</u> (Dimension Axiom)

Let X be a one-point space. Then

$$H^p(X;G) = \begin{cases} G, & \text{if } p = 0 \\ 0, & \text{if } p > 0. \end{cases}$$

<u>proof.</u> The singular chain complex for X is as follows:

$$\begin{array}{ccccccccccc} \cdots \to & S_{n+1}(X) & \xrightarrow{\partial_{n+1}} & S_n(X) & \xrightarrow{\partial_n} & S_{n-1}(X) & \xrightarrow{\partial_{n-1}} & \cdots \to & S_2(X) & \xrightarrow{\cong} & S_1(X) \\ & \cong & & \cong & & \cong & & & \cong & & \cong \\ & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & & \mathbb{Z} & & \mathbb{Z} \end{array}$$

$$\xrightarrow{0} S_0(X) \to 0, \qquad S_0(X) \cong \mathbb{Z}$$

n odd : $\partial_n = 0$

n even : ∂_n is an isomorphism

Applying $\mathrm{Hom}(\ ,G)$ to the end of the singular complex gives

$$0 \xrightarrow{\partial_0^\#} \mathrm{Hom}(S_0(X),G) \xrightarrow{\partial_1^\# = 0} \mathrm{Hom}(S_1(X),G) \to \cdots$$

Since $\partial_0^\# = 0$, it follows that

$$\begin{aligned} H^0(X;G) = \ker \partial_1^\# / \operatorname{im} \partial_0^\# = \ker \partial_1^\# &= \mathrm{Hom}(S_0(X),G) \\ &\cong \mathrm{Hom}(\mathbb{Z},G) \end{aligned}$$

The homomorphism $\mathrm{Hom}(\mathbb{Z},G) \to G$, $f \mapsto f(1)$, is an isomorphism. Thus $H^0(X,G) \cong G$.

The chain complex is

$$\cdots \to \mathrm{Hom}(S_{n-1}(X), G) \xrightarrow{\partial_n^\#} \mathrm{Hom}(S_n(X), G) \xrightarrow{\partial_{n+1}^\#} \mathrm{Hom}(S_{n+1}(X), G)$$
$$\xrightarrow{\partial_{n+2}^\#} \mathrm{Hom}(S_{n+2}(X), G)$$
$$\to \cdots$$

n odd $\Rightarrow \partial_n = 0 \Rightarrow \partial_n^\# = 0$
n even $\Rightarrow \partial_n$ isom $\Rightarrow \partial_n^\#$ isom.

$H^n(X;G) = \ker \partial_{n+1}^\# / \mathrm{im}\, \partial_n^\#$

n odd $\Rightarrow H^n(X;G) = \ker \overbrace{\partial_{n+1}^\#}^{\text{isom}} / \mathrm{im}\, \partial_n^\# = 0/0 = 0$
$0 < n$ even $\Rightarrow H^n(X;G) = \ker \underbrace{\partial_{n+1}^\#}_{0} / \mathrm{im}\, \underbrace{\partial_n^\#}_{\text{surj}} = \mathrm{Hom}(S_n(X), G) / \mathrm{Hom}(S_n(X), G)$
$= 0.$

$\square$

<u>Theorem 13.5</u> (Homotopy Axiom)

Let $f, g: X \to Y$ be homotopic. Then, for all $n \geq 0$, f and g induce the same homomorphisms

$$H^n(Y;G) \to H^n(X;G).$$

<u>proof.</u> This proof follows from the proof of the Homotopy Axiom for homology.

Let $\lambda_i : X \to X \times I$, $x \mapsto (x, i)$, for $i = 0, 1$. We showed that if $H_n(\lambda_0) = H_n(\lambda_1) : H_n(X) \to H_n(X \times I)$, then $H_n(f) = H_n(g)$.

(Let $F: X \times I \to Y$ be a homotopy, $F: f \simeq g$. Then $f = F \circ \lambda_0$, $g = F \circ \lambda_1$. Thus

$$\begin{aligned} H_n(f) = H_n(F \circ \lambda_0) &= H_n(F) \circ H_n(\lambda_0) \\ &= H_n(F) \circ H_n(\lambda_1) \\ &= H_n(F \circ \lambda_1) = H_n(g). \end{aligned}$$)

To show that $H_n(\lambda_0) = H_n(\lambda_1)$ we constructed a chain homotopy $P : S_*(X) \to S_*(X \times I)$, i.e. homomorphisms $P_n : S_n(X) \to S_{n+1}(X \times I)$ satisfying

$$\lambda_{1\#} - \lambda_{0\#} = \partial'_{n+1} P_n + P_{n-1} \partial_n .$$

Applying the functor $\mathrm{Hom}(\ ,G)$ gives the following

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Hom}(S_{n-1}(X\times I), G) \arrow[r, "\partial'^{\#}_n"] \arrow[d, "\lambda_0^{\#},\ \lambda_1^{\#}"'] & \mathrm{Hom}(S_n(X\times I), G) \arrow[r, "\partial'^{\#}_{n+1}"] \arrow[d, "\lambda_0^{\#},\lambda_1^{\#}"'] \arrow[dl, "P^{\#}_{n-1}"'] & \mathrm{Hom}(S_{n+1}(X\times I), G) \arrow[r] \arrow[d, "\lambda_0^{\#},\lambda_1^{\#}"'] \arrow[dl, "P^{\#}_n"'] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}(S_{n-1}(X), G) \arrow[r, "\partial^{\#}_n"'] & \mathrm{Hom}(S_n(X), G) \arrow[r, "\partial^{\#}_{n+1}"'] & \mathrm{Hom}(S_{n+1}(X), G) \arrow[r] & \cdots
\end{tikzcd}$$

Then
$$\begin{aligned}
P_n^{\#} \partial'^{\#}_{n+1} + \partial_n^{\#} P_{n-1}^{\#} &= (\partial'_{n+1} P_n)^{\#} + (P_{n-1}\partial_n)^{\#} \\
&= (\partial'_{n+1} P_n + P_{n-1}\partial_n)^{\#} \\
&= (\lambda_{1\#} - \lambda_{0\#})^{\#} = \lambda_1^{\#} - \lambda_0^{\#} .
\end{aligned}$$

Thus: $H^n(\lambda_1) = H^n(\lambda_0)$. But then

$$\begin{aligned}
H^n(f) = H^n(F \circ \lambda_0) &= H^n(\lambda_0) \circ H^n(F) \\
&= H^n(\lambda_1) \circ H^n(F) \\
&= H^n(F \circ \lambda_1) \\
&= H^n(g). \quad \square
\end{aligned}$$

The next lemma shows that the functor $\mathrm{Hom}(\ ,G)$ is <u>left exact.</u>

<u>Lemma 13.6.</u> Let G be an abelian group, and let

$$A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$$

be an exact sequence of abelian groups. Then there is an exact sequence

$$0 \to \mathrm{Hom}(A'', G) \xrightarrow{p^{\#}} \mathrm{Hom}(A, G) \xrightarrow{i^{\#}} \mathrm{Hom}(A', G).$$

proof.

1) $p^{\#}$ is injective: Let $\varphi : A'' \to G$ be a homomorphism. Assume $0 = p^{\#}(\varphi) = \varphi p$. Then $\varphi|\mathrm{im}\, p = 0$. But p surjective $\Rightarrow \varphi = 0$.

2) $\mathrm{im}\, p^{\#} \subset \ker i^{\#}$: Let $\varphi : A'' \to G$ be a homomorphism. Then $i^{\#} p^{\#} \varphi = \varphi p i = 0$, since $pi = 0$.

3) $\ker i^{\#} \subset \mathrm{im}\, p^{\#}$: Let $g : A \to G$ be a homomorphism. Assume $0 = i^{\#}(g) = gi$. Define

$$\hat{g} : A'' \to G, \quad \hat{g}(a'') = g(a) \text{ if } p(a) = a''.$$

$\hat{g}$ is well defined: Assume $a_0 \in A$, $p(a_0) = a'' = p(a)$. Then

$$p(a - a_0) = p(a) - p(a_0) = a'' - a'' = 0.$$

Then

$$a - a_0 \in \ker p = \mathrm{im}\, i,$$

and $\quad a - a_0 = i(a')$, for some $a' \in A'$. Therefore

$$g(a - a_0) = g(i(a')) = 0, \text{ since } gi = 0.$$

Thus

$$g(a) = g(a_0).$$

But then $p^{\#}(\hat{g}) = \hat{g} p = g$, since $\hat{g}(p(a)) = g(a)$, for all $a \in A$. $\square$

The following example shows that $i^{\#}$ does not need to be surjective even if i is injective:

Example Let $G = \mathbb{Z}$. Consider the exact sequence

$$0 \to \mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Q} \xrightarrow{p} \mathbb{Q}/\mathbb{Z} \to 0,$$

where i is the inclusion and p is the quotient map. Then i is injective and it induces

$$i^{\#}: \mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}), \ f \mapsto fi.$$

Let $f: \mathbb{Q} \to \mathbb{Z}$ be a homom. Assume $f(1) = n$. Then

$$\underbrace{f(\tfrac{1}{m}) + \cdots + f(\tfrac{1}{m})}_{m} = f(m \cdot \tfrac{1}{m}) = f(1) = n$$

$\Rightarrow f(\frac{1}{m}) = \frac{n}{m}$, for all $m \in \mathbb{N}$ $\Rightarrow n = 0$. Thus the only homomorphism $\mathbb{Q} \to \mathbb{Z}$ is trivial, i.e., $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) = 0$. Then $i^{\#}$ can not be surjective.

Corollary 13.7. Let G be an abelian group. Then

1) $\mathrm{Hom}(\mathbb{Z}, G) \cong G$,

2) $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, G) \cong G[m] = \{x \in G \mid mx = 0\}$,

3) $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd\{m, n\}$.

proof.

1) The homomorphism $\varepsilon: \mathrm{Hom}(\mathbb{Z}, G) \to G$, $f \mapsto f(1)$, is an isomorphism.

2) Consider the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/m\mathbb{Z} \to 0, \qquad (*)$$

where m = multiplication by m
p = the quotient map

Apply $\mathrm{Hom}(\ ,G)$ to $(*)$ $\Rightarrow$ obtain an exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, G) \xrightarrow{p^\#} \mathrm{Hom}(\mathbb{Z}, G) \xrightarrow{m^\#} \mathrm{Hom}(\mathbb{Z}, G).$$

Therefore, $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, G) \cong \mathrm{im}\, p^\# = \ker m^\#$. Here, for $x \in \mathbb{Z}$, and for $\varphi \in \mathrm{Hom}(\mathbb{Z}, G)$,

$$(m^\#\varphi)(x) = (\varphi m)(x) = \varphi(mx) = m\varphi(x) = (m\varphi)(x)$$

Thus $m^\#$ is also multiplication by m. The following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, G) \arrow[r, "p^\#"] \arrow[dr, "\varepsilon p^\#"'] & \mathrm{Hom}(\mathbb{Z}, G) \arrow[r, "m^\#"] \arrow[d, "\varepsilon", "\cong"'] & \mathrm{Hom}(\mathbb{Z}, G) \arrow[d, "\varepsilon", "\cong"'] \\
& G \arrow[r, "m"'] & G
\end{tikzcd}$$

$$\begin{tikzcd}
\varphi \arrow[r, mapsto] \arrow[d, mapsto] & m\varphi \arrow[d, mapsto] \\
\varphi(1) \arrow[r, mapsto] & m\varphi(1)
\end{tikzcd}$$

Thus

$$\begin{aligned}
\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, G) &\cong \ker m^\# \\
&= \ker(\varepsilon m^\#) \quad , \varepsilon \text{ isom.} \\
&= \ker(m\varepsilon) \\
&= \varepsilon^{-1} m^{-1}(0) \\
&\cong m^{-1}(0) = \ker m \quad , \varepsilon \text{ isom.} \\
&= \{x \in G \mid mx = 0\} = G[m]
\end{aligned}$$

3) $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \overset{2}{\cong} (\mathbb{Z}/n\mathbb{Z})[m] \cong \mathbb{Z}/d\mathbb{Z}.$

$\square$

Exercise Let $F: Ab \to Ab$ be an additive contravariant functor. Let

$$0 \to A \to B \to C \to 0$$

be a split exact sequence. Then also

$$0 \to FC \to FB \to FA \to 0$$

is a split exact sequence. If I is a finite index set, then

$$F\left(\sum_{i \in I} A_i\right) \cong \sum_{i \in I} F(A_i).$$

Lemma 13.8. Let X be a topological space, and let A be a subspace of X. Let G be an abelian group. For every $n \geq 0$, there is an exact sequence of abelian groups

$$0 \to \mathrm{Hom}(S_n(X)/S_n(A), G) \to \mathrm{Hom}(S_n(X), G) \to \mathrm{Hom}(S_n(A), G) \to 0.$$

Thus there is a short exact sequence of complexes

$$0 \to \mathrm{Hom}(S_*(X)/S_*(A), G) \to \mathrm{Hom}(S_*(X), G) \to \mathrm{Hom}(S_*(A), G) \to 0.$$

proof. Since $S_n(X)/S_n(A)$ is a free abelian group, it follows from what was proved earlier (see the notes about free abelian groups) that the sequence

$$0 \to S_n(A) \hookrightarrow S_n(X) \to S_n(X)/S_n(A) \to 0$$

is a split short exact sequence. By the previous exercise, the sequence remains split exact after applying $\mathrm{Hom}(\ ,G)$. Then also the sequence of complexes is exact. $\square$

Definition 13.9. Let X be a topological space, and let A be a subspace of X. Let G be an abelian group. The n^{th} relative cohomology group with coefficients G is

$$H^n(X,A;G) = H_{-n}\,\mathrm{Hom}(S_*(X)/S_*(A), G).$$

$$= \ker(\bar{\partial}_{n+1})^\# / \mathrm{im}(\bar{\partial}_n)^\#,$$

where

$$\bar{\partial}_{n+1} : S_{n+1}(X)/S_{n+1}(A) \longrightarrow S_n(X)/S_n(A),$$

$$c + S_{n+1}(A) \longmapsto \partial_{n+1}c + S_n(A).$$

Since there is a short exact sequence of complexes

$$0 \to \mathrm{Hom}(S_*(X)/S_*(A), G) \longrightarrow \mathrm{Hom}(S_*(X), G) \longrightarrow \mathrm{Hom}(S_*(A), G) \to 0,$$

there is a connecting homomorphism

$$d : H^n(A;G) \longrightarrow H^{n+1}(X,A;G).$$

Thus we obtain a long exact sequence in cohomology:

Theorem 13.10. (Long Exact Sequence)

Let X be a topological space, and let A be a subspace of X. Let G be an abelian group. Then there is an exact sequence

$$0 \to H^0(X,A;G) \to H^0(X;G) \to H^0(A;G) \xrightarrow{d} H^1(X,A;G)$$

$$\to H^1(X;G) \to \dots$$

The connecting homomorphisms are natural. $\square$

Naturality of the connecting homomorphism means the following: Let $f:(X,A)\to(Y,B)$. Then there is a commutative diagram

$$\begin{array}{ccccccccccc}
0 \to & H^0(Y,B;G) & \to & H^0(Y;G) & \to & H^0(B;G) & \xrightarrow{d} & H^1(Y,B;G) & \to & H^1(Y;G) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^0(X,A;G) & \to & H^0(X;G) & \to & H^0(A;G) & \xrightarrow{d} & H^1(X,A;G) & \to & H^1(X;G) & \to \cdots
\end{array}$$

where the vertical maps are induced by f.

Theorem 13.11. (Excision)

Let X be a topological space, and let X_1 and X_2 be subspaces of X with $X=\mathring{X}_1\cup\mathring{X}_2$ ($\mathring{X}_i$ = the interior of X_i, for $i=1,2$). Let G be an abelian group. Then the inclusion $j:(X_1, X_1\cap X_2)\to(X,X_2)$ induces isomorphisms for all $n\geq 0$,

$$j^*: H^n(X,X_2;G)\to H^n(X_1, X_1\cap X_2;G).$$

proof. Similar to the corresponding proof for homology. $\square$

14. Universal Coefficients Theorems for Cohomology

For cohomology, there are two universal coefficients theorems:

1) How $H^n(X;G)$ is determined by $H_*(X)$.

2) How $H^n(X;G)$ is determined by $H^*(X)=H^*(X;\mathbb{Z})$.

Definition 14.1. Let A and G be abelian groups. Choose an exact sequence

$$0 \to R \xhookrightarrow{i} F \xrightarrow{p} A \to 0,$$

where F and R are free abelian. Then there is an exact sequence

$$0 \to \mathrm{Hom}(A,G) \xrightarrow{p^\#} \mathrm{Hom}(F,G) \xrightarrow{i^\#} \mathrm{Hom}(R,G)$$

By definition, the cokernel of $i^\#$ is

$$\mathrm{coker}\, i^\# = \mathrm{Hom}(R,G) / i^\# \mathrm{Hom}(F,G).$$

We define : $\mathrm{Ext}(A,G) = \mathrm{coker}\, i^\#$.

It can be shown that Ext is independent of the choice of the exact sequence $0 \to R \hookrightarrow F \to A \to 0$.

Definition 14.2. An abelian group G is called divisible, if for every $x \in G$ and for every $n \in \mathbb{N}$, there is $y \in G$: $ny = x$.

Example The groups $\mathbb{Q}, \mathbb{R}, \mathbb{C}, S^1, \mathbb{Q}/\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$ are divisible.

Here are the basic properties of Ext. The proofs can be found in any book on homological algebra:

1. **Ext1:** Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence of abelian groups. Then there is an exact sequence

$$0 \to \mathrm{Hom}(A'',G) \to \mathrm{Hom}(A,G) \to \mathrm{Hom}(A',G) \to$$

$$\mathrm{Ext}(A'',G) \to \mathrm{Ext}(A,G) \to \mathrm{Ext}(A',G) \to 0.$$

1'. Ext 1': Let $0 \to G' \to G \to G'' \to 0$ be a short exact sequence of abelian groups. Let A be an abelian group. Then there is an exact sequence

$$0 \to \operatorname{Hom}(A, G') \to \operatorname{Hom}(A, G) \to \operatorname{Hom}(A, G'') \to$$

$$\operatorname{Ext}(A, G') \to \operatorname{Ext}(A, G) \to \operatorname{Ext}(A, G'') \to 0.$$

2. Ext 2: If F is a free abelian group, then

$$\operatorname{Ext}(F, G) = 0.$$

2'. Ext 2': If D is a divisible group, then

$$\operatorname{Ext}(A, D) = 0.$$

Let $\{A_j \mid j \in J\}$ be a family of abelian groups. We denote by $\prod A_j$ the abelian group whose elements are all J-tuples (a_j), the group operation on $\prod A_j$ is the coordinatewise addition. Then $\sum A_j$ is the subgroup of $\prod A_j$ consisting off all J-tuples (a_j) having only finitely many nonzero coordinates. For finite J, $\sum A_j = \prod A_j$.

3. Ext 3: $\operatorname{Ext}(\sum A_j, G) \cong \prod \operatorname{Ext}(A_j, G)$.

3'. Ext 3': $\operatorname{Ext}(A, \prod G_j) \cong \prod \operatorname{Ext}(A, G_j)$.

4. Ext 4: $\operatorname{Ext}(\mathbb{Z}/m\mathbb{Z}, G) \cong G/mG$.

Example: $\mathbb{Z}$ free abelian $\overset{\text{Ext 2}}{\Longrightarrow}$ $\operatorname{Ext}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = 0$.

Ext 4 $\Rightarrow$ $\operatorname{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$.

Then $\operatorname{Ext}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \not\cong \operatorname{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$.

Let A be an abelian group. Let $0 \to R \xrightarrow{i} F \xrightarrow{p} A \to 0$ be an exact sequence with F (and also R) free abelian. Then Ext 2 $\Rightarrow$ $\text{Ext}(F,G) = 0$. The exact sequence

$$0 \to \text{Hom}(A,G) \xrightarrow{p^\#} \text{Hom}(F,G) \xrightarrow{i^\#} \text{Hom}(R,G) \xrightarrow{\phi} \text{Ext}(A,G)$$
$$\to \text{Ext}(F,G) \to \text{Ext}(R,G) \to 0$$

becomes

$$0 \to \text{Hom}(A,G) \to \text{Hom}(F,G) \to \text{Hom}(R,G) \to \text{Ext}(A,G)$$
$$\to 0 \to 0 \to 0$$

Then $$\text{Ext}(A,G) = \text{im}\,\phi \cong \text{Hom}(R,G)/\ker\phi$$
$$= \text{Hom}(R,G)/\text{im}\, i^\#$$
$$= \text{coker}\, i^\#.$$

Thus we ended up having the definition of Ext.

We next construct a homomorphism

$$\beta : H^n(\text{Hom}(S_*, G)) \to \text{Hom}(H_n(S_*), G),$$

where S_* is a chain complex, and G is an abelian group.

1) Let φ be an n-cocycle. Then $\varphi \in \text{Hom}(S_n, G)$, and $0 = \delta\varphi = \varphi\partial_{n+1}$. Then $\varphi(\underbrace{\partial_{n+1}x}_{\in B_n}) = 0$ $\forall x \in S_{n+1}$, i.e., $\varphi|B_n = 0$.
$\Rightarrow$ φ induces a homomorphism $S_n/B_n \to G$
$\Rightarrow$ — " — $\varphi' : H_n(S_*) = Z_n/B_n \to G$,
$z_n + B_n \mapsto \varphi(z_n)$.

2) Assume that φ is an n-coboundary. Then $\varphi \in Hom(S_n, G)$, $\varphi = \delta\psi = \psi\partial_n$, for some $\psi \in Hom(S_{n-1}, G)$. Since φ is an n-cocycle, it induces a homomorphism

$$\varphi' : Z_n/B_n \longrightarrow G, \quad z_n + B_n \mapsto \varphi(z_n) = \psi \underbrace{\partial_n z_n}_{=0} = 0.$$

Let then

$$\beta : H^n(Hom(S_*, G)) \longrightarrow Hom(H_n(S_*), G),$$

$$[\varphi] \mapsto \varphi', \quad \text{where } \varphi'(z_n + B_n) = \varphi(z_n).$$

By 1 and 2, β is a well-defined homomorphism.

<u>Theorem 14.3.</u> (Dual Universal Coefficients)

Let X be a topological space, and let G be an abelian group.

1) For every $n \geq 0$, there is an exact sequence

$$0 \to Ext(H_{n-1}(X), G) \to H^n(X; G) \xrightarrow{\beta} Hom(H_n(X), G) \to 0, \quad (*)$$

where β is as above.

2) The sequence $(*)$ splits: For every $n \geq 0$, there is an isomorphism

$$H^n(X; G) \cong Hom(H_n(X), G) \oplus Ext(H_{n-1}(X), G).$$

<u>Proof.</u> Just like Universal Coefficients Theorem for Homology, this result can be proved by proving a more general result: Let (C_*, ∂) be a free chain complex. Then

$$H^n(\mathrm{Hom}(C_*, G)) \cong \mathrm{Hom}(H_n(C_*), G) \oplus \mathrm{Ext}(H_{n-1}(X), G).$$

Choosing $C_* = S_*(X)$ gives the result of the theorem. The more general claim can be proved by following the proof of the Universal Coefficient Theorem for Homology, $- \otimes G$ should be replaced by $\mathrm{Hom}(\ , G)$, Tor1 and Tor2 should be replaced by Ext1 and Ext 2, respectively. □

<u>Corollary 14.4.</u> Let F be a field of characteristic zero (for example, $F = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$). Then, for every $n \geq 0$,

$$H^n(X; F) \cong \mathrm{Hom}(H_n(X), F).$$

<u>proof.</u> A field of characteristic zero is divisible when considered as an additive group. Thus Ext2' $\Rightarrow \mathrm{Ext}(H_{n-1}(X), F) = 0$. □

The <u>integral cohomology</u> $H^*(X; \mathbb{Z})$ is usually denoted by $H^*(X)$.

<u>Definition 14.5.</u> Let C_* be a chain complex. If every C_n is finitely generated (f.g.), then C_* is said to be of <u>finite type.</u> A topological space X is of <u>finite type</u>, if $H_n(X)$ is finitely generated, for every n.

Every compact CW complex is of finite type.

<u>Lemma 14.6.</u> Let X be a topological space of finite type. Then there exists a free chain complex C_* of finite type such that C_* is chain equivalent to $S_*(X)$.

proof. Let $\nu_n : Z_n(X) \to Z_n(X)/B_n(X) = H_n(X)$ be the quotient map. Since $H_n(X)$ is f.g, $Z_n(X)$ has a f.g. subgroup F_n s.t. $\nu_n|F_n : F_n \to H_n(X)$ is surjective. $Z_n(X)$ free abelian $\Rightarrow$ F_n free abelian. Let $F_n' = \ker(\nu_n|F_n)$. Let

$$C_n = F_n \oplus F'_{n-1},$$

and let $\quad d_n : C_n \to C_{n-1},\ (\alpha,\alpha') \mapsto (\alpha', 0),$

where $\alpha \in F_n$, $\alpha' \in F'_{n-1}$. For every n, C_n is a f.g. free abelian group, and

$$H_n(C_*) = \ker d_n / \operatorname{im} d_{n+1} = F_n/F_n' = H_n(X).$$

We next construct a chain map $f : C_* \to S_*(X)$:

$$\begin{tikzcd} & F_n' \subset B_n(X) \arrow[dl, dashed, "h_n"'] \arrow[d, hook, "i"] \\ S_{n+1}(X) \arrow[r, "\partial_{n+1}"'] & S_n(X) \end{tikzcd} \qquad i = \text{inclusion}$$

$F_n' \subset B_n(X)$, since $\nu|F_n' = 0$.
Thus $i(F_n') = F_n' \subset \partial_{n+1}(S_{n+1}(X))$
F_n' free abelian $\Rightarrow$ $\exists$ homom. $h_n : F_n' \to S_{n+1}(X)$ s.t. $\partial_{n+1} h_n = i$.

Let $\quad f_n : C_n \to S_n(X),\ (\alpha,\alpha') \mapsto \alpha + h_{n-1}(\alpha'),$

where $\alpha \in F_n$, $\alpha' \in F_n'$. Then

$$\partial_n f_n(\alpha,\alpha') = \partial_n(\alpha + h_{n-1}(\alpha')) = \partial_n \alpha + \overbrace{\partial_n h_{n-1}}^{i}(\alpha')$$
$$= 0 + \alpha' = \alpha',$$

since $\alpha \in F_n \subset B_n(X)$.

Also, $f_{n-1} d_n(\alpha, \alpha') = f_{n-1}(\alpha', 0) = \alpha' + h_{n-2}(0) = \alpha'$.

Thus $\partial_n f_n = f_{n-1} d_n$.

$\Rightarrow f_n$ is a chain map.

Let (α, α') be an n-cycle of C_*. Then $d_n(\alpha, \alpha') = (\alpha', 0) = 0$, and must be $\alpha' = 0$. $\Rightarrow h_{n-1}(\alpha') = 0$. Then

$$f_* : H_n(C_*) \longrightarrow H_n(X), \quad [(\alpha, 0)] \longmapsto [f_n(\alpha, 0)] = [\alpha],$$

is an isomorphism for every $n \geq 0$. Since C_* and $S_*(X)$ are free chain complexes, it follows from Theorem 8.13, that f is a chain equivalence.

$\square$

Lemma 14.7. Let C_* be a free chain complex of finite type. Let G be an abelian group. Then the chain complexes

$$\mathrm{Hom}(C_*, \mathbb{Z}) \otimes G \quad \text{and} \quad \mathrm{Hom}(C_*, G)$$

are isomorphic.

Proof. For every n, let

$$\mu_n : \mathrm{Hom}(C_n, \mathbb{Z}) \otimes G \longrightarrow \mathrm{Hom}(C_n, G),$$

$$\mu_n(f \otimes g) : C_n \to G, \quad c \mapsto f(c) g.$$

The definition makes sense, since $f(c) \in \mathbb{Z}$.
For $c, c' \in C_n$,

$$c + c' \longmapsto f(c + c') g = [f(c) + f(c')] g$$
$$= f(c) g + f(c') g$$

$\therefore$ $\mu_n(f \otimes g)$ is a homomorphism

We next show that μ_n isomorphism. This is done by induction on the number of generators of C_n.

First, assume C_n has 1 generator. Then $C_n \cong \mathbb{Z}$, and we have

$$\mu_n : \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \otimes G \to \mathrm{Hom}(\mathbb{Z}, G).$$

Notice that every homomorphism $\mathbb{Z} \to \mathbb{Z}$ is of the form $\phi_m : z \mapsto mz$, for some $m \in \mathbb{Z}$. Moreover, $\phi_m = m \cdot id$, where $id : \mathbb{Z} \to \mathbb{Z}$.

μ_n is surjective: Let $h : \mathbb{Z} \to G$ be a homomorphism. Let $h(1) = g_1$. Then $h : z \mapsto z g_1$. Now

$$\mu_n(id \otimes g_1) : \mathbb{Z} \to G, \; z \mapsto id(z) g_1 = z g_1.$$

Thus $\mu_n(id \otimes g_1) = h$. $\therefore$ μ_n is surjective

μ_n is injective: Notice that for all $g \in G$ and for all $m \in \mathbb{Z}$,

$$\phi_m \otimes g = (m \cdot id) \otimes g = id \otimes mg.$$

Assume $\mu_n(\phi_m \otimes g) = \mu_n(\phi_{m'} \otimes g')$, i.e. that

$$\mu_n(id \otimes mg) = \mu_n(id \otimes m'g').$$

Let $z \in \mathbb{Z}$. Then $zmg = zm'g'$, since $\mu_n(id \otimes mg) = \mu_n(id \otimes m'g')$.

$\Rightarrow mg = m'g' \Rightarrow \phi_m \otimes g = id \otimes mg = id \otimes m'g' = \phi_{m'} \otimes g'$.

$\therefore$ μ_n is injective

$\therefore$ μ_n is an isomorphism.

Assume then that μ_n is an isomorphism, if the rank of C_n is $\leq m$. Check that μ_n is an isom. also if the rank of C_n equals $m+1$. Thus, assume the rank is $m+1$. Write $C_n = C'_n \oplus \mathbb{Z}$, where the rank of C'_n is m. Denote

$$\mu_{C_n} = \mu_n : \mathrm{Hom}(C_n, \mathbb{Z}) \otimes G \to \mathrm{Hom}(C_n, G)$$

and similarly,

$$\mu_{C'_n} = \mu_n : \mathrm{Hom}(C'_n, \mathbb{Z}) \otimes G \to \mathrm{Hom}(C'_n, G)$$

$$\mu_{\mathbb{Z}} = \mu_n : \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \otimes G \to \mathrm{Hom}(\mathbb{Z}, G).$$

By induction, $\mu_{C'_n}$ and $\mu_{\mathbb{Z}}$ are isomorphisms.

The diagram

$$\begin{tikzcd}
\mathrm{Hom}(C_n, \mathbb{Z}) \otimes G \arrow[r, "\cong"] \arrow[d, "\mu_n"'] & (\mathrm{Hom}(C'_n, \mathbb{Z}) \otimes G) \oplus (\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \otimes G) \arrow[d, "\mu_{C'_n} \oplus \mu_{\mathbb{Z}}", "\cong"'] \\
\mathrm{Hom}(\underset{C'_n \oplus \mathbb{Z}}{\overset{}{C_n}}, G) \arrow[r, "\cong"'] & \mathrm{Hom}(C'_n, G) \oplus \mathrm{Hom}(\mathbb{Z}, G)
\end{tikzcd}$$

commutes. Thus μ_n is an isomorphism. The horizontal isomorphisms are of the following form:

$$\begin{aligned} \mathrm{Hom}(A \oplus B, G) &\xrightarrow{\cong} \mathrm{Hom}(A, G) \oplus \mathrm{Hom}(B, G), \\ \phi &\mapsto (\phi|A, \phi|B) \end{aligned}$$

where A and B are abelian groups.
Also, $(A \oplus B) \otimes G \cong (A \otimes G) \oplus (B \otimes G)$.

It remains to check that μ is a chain map.

Consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}(C_n, \mathbb{Z}) \otimes G \arrow[r, "\partial_{n+1}^{\#} \otimes id"] \arrow[d, "\mu_n"'] & \mathrm{Hom}(C_{n+1}, \mathbb{Z}) \otimes G \arrow[d, "\mu_{n+1}"] \\
\mathrm{Hom}(C_n, G) \arrow[r, "\partial_{n+1}"'] & \mathrm{Hom}(C_{n+1}, G)
\end{tikzcd}$$

Here

$$\begin{tikzcd}
\phi \otimes g \arrow[r] \arrow[d] & (\phi \partial_{n+1}) \otimes g \arrow[d] \\
\mu_n(\phi \otimes g) \arrow[dr] & \mu_{n+1}((\phi \partial_{n+1}) \otimes g) \\
& (\mu_n(\phi \otimes g)) \circ \partial_{n+1}
\end{tikzcd}$$

and

$$(\mu_n(\phi \otimes g)) \circ \partial_{n+1} : c \mapsto \partial_{n+1} c \mapsto \phi(\partial_{n+1} c) g$$

$$\mu_{n+1}((\phi \partial_{n+1}) \otimes g) : c \mapsto (\phi \partial_{n+1}(c)) g = \phi(\partial_{n+1} c) g$$

Thus $\mu_n(\phi \otimes g) \circ \partial_{n+1} = \mu_{n+1}((\phi \partial_{n+1}) \otimes g)$.

$\Rightarrow$ The diagram commutes.

$\Rightarrow$ μ is a chain map. $\square$