Algebraic Topology II

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Lectures: Tue : 14:15 - 16:00 in B321
Thu : 14:15 - 16:00 in C122

Exercises: Wed : 12:15 - 14:00 in C122

- The first meeting will be on Wed, Jan 27.

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Topics:

- CW-complexes
- cellular homology
- natural transformations
- universal coefficients for homology
- Künneth formula
- cohomology
  - cohomology groups
  - universal coefficients
  - cohomology rings
  - applications

Notation:

$\mathbb{Z}$ = integers
$\mathbb{Q}$ = rational numbers
$\mathbb{R}$ = real numbers
$\mathbb{C}$ = complex numbers
$I = [0,1]$, the closed unit interval
$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R} \text{ for all } i\}$
$I^n = \{(x_1, \dots, x_n) \mid x_i \in I \text{ for all } i\}$

$x \in \mathbb{R}$: the absolute value of x is $|x|$

$x \in \mathbb{R}^n$: the norm of x is $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$, where $x = (x_1, \dots, x_n)$.

$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$, the unit sphere, also called the n-sphere of radius 1 and center the origin.

$S^0 = \{x \in \mathbb{R} \mid |x| = 1\} = \{-1, 1\}$

$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, the closed n-disk, also called the closed n-ball.

$D^0 = \{0\}$

1. Quotient Spaces and Attaching Cells

Let X be a topological space and let $\sim$ be an equivalence relation on X. Let $[x]$ denote the equivalence class of $x \in X$, and let $X/\sim$ denote the set of equivalence classes. Let $\nu : X \to X/\sim$, $x \mapsto [x]$. Then ν is a surjection. Define a topology on $X/\sim$ by requiring the following: A subset U of $X/\sim$ is open if and only if the inverse image $\nu^{-1}(U)$ is open in X. Then ν becomes a continuous map. The space $X/\sim$ is called a quotient space and the topology on $X/\sim$ is called the quotient topology defined by ν.

Let $A \subset X$. Identifying all points in A with each others, and no other points, defines an equivalence relation on X. The quotient space obtained is denoted by X/A.

Example 1.1. Let $X = I = [0,1]$ and let $A = [0,1)$. Let $\nu : X \to X/A$ be the quotient map. The quotient space X/A has two elements: $[0]$ and $[1]$. The inverse image $\nu^{-1}([1]) = \{1\}$ is not open in X. Thus $\{[1]\}$ is not open in X/A. Therefore, $\{[0]\} = (X/A) - \{[1]\}$ is not closed in X/A. Thus X/A is not a Hausdorff space.

Example 1.1. shows that a quotient space of a Hausdorff space does not need to be Hausdorff.

Let $\sim$ be an equivalence relation on a topological space X. The graph of $\sim$ is

$$G = \{(x_1, x_2) \in X \times X \mid x_1 \sim x_2\}.$$

Let $\nu : X \to X/\sim$ be the quotient map and let

$$D = \{([x],[x]) \in X/\sim \times X/\sim \mid x \in X\}$$

denote the diagonal of $X/\sim$. The diagonal D is a closed subset of $X/\sim \times X/\sim$, if and only if $X/\sim$ is a Hausdorff space. Assume $X/\sim$ is a Hausdorff space. Then the graph

$$G = (\nu \times \nu)^{-1}(D),$$

where $\nu \times \nu : X \times X \to X/\sim \times X/\sim, \ (x,y) \mapsto ([x],[y]).$

Since $\nu \times \nu$ is continuous, it follows that the graph G is closed in $X \times X$.

<u>Lemma 1.2.</u> Let W and Z be topological spaces, and let $\nu : W \to Z$ be a closed map. Let $S \subset Z$ and let U be an open subset of W with $\nu^{-1}(S) \subset U$. Then there is an open subset V of Z with

$$S \subset V \quad \text{and} \quad \nu^{-1}(V) \subset U.$$

<u>proof.</u> Let $V = Z - \nu(W-U)$. Then U open in W $\Rightarrow$ $W-U$ closed, and ν closed $\Rightarrow$ $\nu(W-U)$ closed $\Rightarrow$ $V = Z - \nu(W-U)$ open. Since $\nu^{-1}(S) \subset U$, it follows that $S \cap \nu(W-U) = \emptyset$. Thus $S \subset Z - \nu(W-U) = V$. Also,

$$\nu^{-1}(V) = \nu^{-1}(Z - \nu(W-U)) = W - \nu^{-1}(\nu(W-U))$$
$$\subset W - (W-U) = U. \qquad \square$$

<u>Theorem 1.3.</u> Let X be a compact Hausdorff space, and let $\sim$ be an equivalence relation on X. Assume the graph G of $\sim$ is closed in $X \times X$. Then the quotient space $X/\sim$ is a compact Hausdorff space.

Proof. Let $\nu : X \to X/\sim$ be the quotient map. Since X is compact and ν is continuous, it follows that $X/\sim$ is compact.

Let

$$p_i : X \times X \to X, \ (x_1, x_2) \mapsto x_i, \quad \text{for } i = 1, 2,$$

be the projection. Let $C \subset X$. Then

$$\begin{aligned} p_2(p_1^{-1}(C) \cap G) &= p_2((C \times X) \cap G) \\ &= \{ y \in X \mid y \sim x \text{ for some } x \in C \} \\ &= \nu^{-1}(\nu(C)). \end{aligned}$$

Assume then that C is closed in X. Then:

p_1 continuous $\Rightarrow$ $p_1^{-1}(C)$ is closed in $X \times X$,

G closed $\Rightarrow$ $p_1^{-1}(C) \cap G$ is closed in $X \times X$,

X compact Hausdorff $\Rightarrow$ $p_2 : X \times X \to X$ is a closed map.

Then $\nu^{-1}(\nu(C))$ is closed in X. Hence

$$\nu^{-1}(X/\sim - \nu(C)) = X - \nu^{-1}(\nu(C))$$

is open in X and, consequently, $X/\sim - \nu(C)$ is open in $X/\sim$. Therefore $\nu(C)$ is closed in $X/\sim$. It follows that ν is a closed map.

It follows that every point in $X/\sim$ is closed. Assume $[x], [y] \in X/\sim$, $[x] \neq [y]$. Then $\nu^{-1}([x])$ and $\nu^{-1}([y])$ are closed in X, $\nu^{-1}([x]) \cap \nu^{-1}([y]) = \emptyset$. Now, X compact Hausdorff $\Rightarrow$ X is normal. Then $\exists$ open subsets $U_{[x]}$ and $U_{[y]}$ of X : $\nu^{-1}([x]) \subset U_{[x]}$, $\nu^{-1}([y]) \subset U_{[y]}$, $U_{[x]} \cap U_{[y]} = \emptyset$. Lemma 1.2 $\Rightarrow$ $\exists$ open subsets $V_{[x]}$ and $V_{[y]}$ of $X/\sim$: $[x] \in V_{[x]}$, $[y] \in V_{[y]}$, $\nu^{-1}(V_{[x]}) \subset U_{[x]}$, $\nu^{-1}(V_{[y]}) \subset U_{[y]}$. Then $V_{[x]} \cap V_{[y]} = \emptyset$, which implies that $X/\sim$ is a Hausdorff space. $\square$

Corollary 1.4. Let X be a compact Hausdorff space, and let A be a closed subset of X. Then X/A is a compact Hausdorff space.

proof. Let $\sim$ be the corresponding equivalence relation on X. Let D be the diagonal of X. The graph of $\sim$ is then

$$G = (A \times A) \cup D.$$

Since X is Hausdorff, D is closed in $X \times X$. Also, $A \times A$ is closed in $X \times X$. Thus G is closed in $X \times X$. The claim now follows from Theorem 1.3. $\square$

Let $\mathbb{H}$ denote the set of quaternions. Then $\mathbb{H}$ is a four-dimensional real vector space with basis $\{1, i, j, k\}$, where

$$\begin{aligned} i^2 = j^2 &= k^2 = -1, \\ ij = -ji &= k \\ jk = -kj &= i \\ ki = -ik &= j. \end{aligned}$$

The vector-space $\mathbb{H}$ has a norm $|\;|$: The norm of $w = a + bi + cj + dk \in \mathbb{H}$ is

$$|w| = |a + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Let $\bar{w} = a - bi - cj - dk$. Then

$$w\bar{w} = |w|^2.$$

Thus, for $w \neq 0$, we define

$$w^{-1} = \frac{1}{|w|^2}\bar{w}.$$

It follows that $\mathbb{H}$ is a division ring. Notice that multiplication on $\mathbb{H}$ is not commutative, since for example $ij \neq ji$.

<u>Definition 1.5.</u> Let F be either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $n \geq 0$. Let F^{n+1} consist of all $(n+1)$-tuples $(x_0, x_1, \ldots, x_n)$, where $x_i \in F$ for all i. Define an equivalence relation $\sim$ on $F^{n+1} - \{0\}$ by $x \sim y$ if there exists $\lambda \in F - \{0\}$ with $x = \lambda y$. The quotient space

$$FP^n = (F^{n+1} - \{0\})/\sim$$

is called <u>F-projective n-space.</u> Then

$\mathbb{R}P^n$ = real projective n-space
$\mathbb{C}P^n$ = complex projective n-space
$\mathbb{H}P^n$ = quaternionic projective n-space.

Denote the equivalence class of $x = (x_0, \ldots, x_n)$ by $[x]$.
For every $n \geq 0$, there is an embedding

$$FP^n \to FP^{n+1}, \quad [x_0, \ldots, x_n] \mapsto [x_0, \ldots, x_n, 0].$$

<u>Some exercises:</u>

<u>Exercise 1.6.</u> For each $n \geq 0$, define an equivalence relation on S^n by $x \sim y$ if $x = \pm y$. Prove that $S^n/\sim$ is homeomorphic to $\mathbb{R}P^n$.

<u>Exercise 1.7.</u> For each $n \geq 0$, define an equivalence relation on S^{2n+1} by $x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{C}$ s.t. $|\lambda| = 1$. Prove that $S^{2n+1}/\sim$ is homeomorphic to $\mathbb{C}P^n$. (Hint: Write $x = (x_1, x_2, \ldots, x_{2n+2}) \in S^{2n+1}$ as an n-tuple of complex numbers: $x = (z_1, \ldots, z_{n+1})$; then $x = \lambda y$ implies $|x| = |\lambda||y|$ and $|\lambda| = 1$.)

<u>Exercise 1.8.</u> For each $n \geq 0$, define an equivalence relation on S^{4n+3} by $x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{H}$ s.t. $|\lambda| = 1$. Prove that $S^{4n+3}/\sim$ is homeomorphic to $\mathbb{H}P^n$. (Hint: If $x, y \in S^{4n+3} \subset \mathbb{H} - \{0\}$, then $x = \lambda y$ implies $|\lambda| = 1$.)

Disjoint Union of topological spaces

Disjoint union of sets:

Let X_j, $j \in J$, be sets, where J is an indexing set. Assume $J \neq \emptyset$, $X_j \neq \emptyset$ $\forall j \in J$. The disjoint union of the X_j, $j \in J$, is

$$\coprod_{j \in J} X_j = \bigcup_{j \in J} U_j,$$

where $U_j = X_j \times \{j\}$ $\forall j \in J$.

Disjoint union of topological spaces:

Let X_j, $j \in J$, be topological spaces. Assume $J \neq \emptyset$ and $X_j \neq \emptyset$ $\forall j \in J$. Let

$$X = \coprod_{j \in J} X_j.$$

Define the canonical injection

$$\varphi_j : X_j \to X, \quad x \mapsto (x, j).$$

We topologize X as follows: Let $O \subset X$. Then O is open in X if and only if $\varphi_j^{-1}(O)$ is open in X_j for every $j \in J$. This topology is called the <u>disjoint union topology</u> on X.

Let $i \in J$ and let U be open in X_i. Then $\varphi_i(U) = U \times \{i\}$, $\varphi_i^{-1}(\varphi_i(U)) = U$ and $\varphi_j^{-1}(\varphi_i(U)) = \varphi_j^{-1}(U \times \{i\}) = \emptyset$ $\forall j \in J - \{i\}$. Thus $\varphi_i(U)$ is open in X. It follows that φ_j is a homeomorphism from X_j to $\varphi_j(X_j)$ $\forall j$. Thus we may identify X_j with the image $\varphi_j(X_j) = X_j \times \{j\}$ in X.

<u>Theorem</u> Let X_j, $j \in J$, and Y be topological spaces. Let

$$f : \coprod_{j \in J} X_j \to Y$$

be a function. Then f is continuous if and only if $f \circ \varphi_j$ is continuous for every $j \in J$, where

$$\varphi_j : X_j \to \coprod_{j \in J} X_j, \quad x \mapsto (x, j).$$

<u>proof.</u> Assume f is continuous. Let O be open in Y. Then $f^{-1}(O)$ is open in $\coprod_{j \in J} X_j$.
Thus

$$(f \circ \varphi_j)^{-1}(O) = \varphi_j^{-1}(f^{-1}(O))$$

is open in X_j, for all $j \in J$. Therefore $f \circ \varphi_j$ is continuous for all $j \in J$.

Conversely, assume that $f \circ \varphi_j : X_j \to Y$ is continuous for every $j \in J$. Let O be open in Y. Then

$$\varphi_j^{-1}(f^{-1}(O)) = (f \circ \varphi_j)^{-1}(O)$$

is open in X_j, for all $j \in J$. By the definition of the disjoint union topology, $f^{-1}(O)$ is open in $\coprod_{j \in J} X_j$. Thus f is continuous. $\square$

Let A_1, A_2, B be topological spaces and let $f_1 : B \to A_1$ and $f_2 : B \to A_2$ be continuous functions.

$$\begin{tikzcd} B \arrow[r, "f_1"] \arrow[d, "f_2"'] & A_1 \\ A_2 & \end{tikzcd} \qquad (*)$$

A solution of the diagram $(*)$ is a topological space C and continuous functions g_1 and g_2 such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "f_1"] \arrow[d, "f_2"'] & A_1 \arrow[d, "g_1"] \\ A_2 \arrow[r, "g_2"'] & C \end{tikzcd}$$

A <u>pushout</u> of the diagram $(*)$ is a solution (C, g_1, g_2) such that for any other solution (D, h_1, h_2) of $(*)$ there exist a unique continuous function $\varphi : C \to D$ making the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "f_1"] \arrow[d, "f_2"'] & A_1 \arrow[d, "g_1"'] \arrow[ddr, "h_1"] & \\ A_2 \arrow[r, "g_2"] \arrow[drr, "h_2"'] & C \arrow[dr, "\varphi"] & \\ & & D \end{tikzcd}$$

<u>Exercise</u> Show that the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "i"'] & Y \arrow[d, "\phi_Y"] \\ X \arrow[r, "\phi_X"'] & X \amalg_\phi Y \end{tikzcd}$$

is a pushout. (Here X and Y are topological spaces, A is a closed subset of Y, $\phi : A \to Y$ is continuous, $i : A \hookrightarrow X$ is the inclusion and ϕ_X and ϕ_Y are characteristic maps.)

Theorem 1.9. For every $n \geq 0$, the projective spaces $\mathbb{R}P^n$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$ are compact and Hausdorff.

Proof.

$\mathbb{R}P^n$: Ex. 1.6 $\Rightarrow$ $\mathbb{R}P^n \approx S^n/\sim$. Here $x, y \in S^n$, $x \sim y$ if $x = \pm y$. Thus the graph of $\sim$ is

$$G = D \cup \{(x, -x) \mid x \in S^n\}$$

is closed in $S^n \times S^n$. Theorem 1.3 $\Rightarrow$ $\mathbb{R}P^n$ is compact Hausdorff.

$\mathbb{C}P^n$: Ex. 1.7 $\Rightarrow$ $\mathbb{C}P^n \approx S^{2n+1}/\sim$. Here $x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Then the graph of $\sim$ is

$$G = \{(x, \lambda x) \mid x \in S^{2n+1}, \lambda \in S^1\}.$$

Let

$$f : S^{2n+1} \times S^1 \to S^{2n+1} \times S^{2n+1}, \quad (x, \lambda) \mapsto (x, \lambda x).$$

Then $G = f(S^{2n+1} \times S^1)$ is compact, which implies that it is also closed. Theorem 1.3 $\Rightarrow$ $\mathbb{C}P^n$ is compact Hausdorff.

$\mathbb{H}P^n$: Similarly, by using Ex. 1.8. $\square$

2. Attaching Cells

Let X_1 and X_2 be topological spaces. The disjoint union of X_1 and X_2 is denoted by

$$X_1 \amalg X_2$$

and called the <u>coproduct</u> of X_1 and X_2.

Then X_1 and X_2 are open subsets of $X_1 \amalg X_2$. Let Y be a topological space, and let $f_i : X_i \to Y$ be continuous, for $i = 1, 2$. Then the map

$$f_1 \amalg f_2 : X_1 \amalg X_2 \to Y,$$

$$(f_1 \amalg f_2)(x) = f_i(x) \text{ for } x \in X_i, \text{ is continuous.}$$

<u>Definition 2.1.</u> Let X and Y be topological spaces, let A be a closed subset of X, and let $f : A \to Y$ be continuous. Identifying a with $f(a)$, for all $a \in A$, and no other points, defines an equivalence relation $\sim$ on $X \amalg Y$. The quotient

$$X \amalg_f Y = (X \amalg Y)/\sim$$

is called an <u>adjunction space</u> (also an attaching space) and the map f is called an <u>attaching map.</u>

<u>Exercise 2.2.</u> Let X and Y be Hausdorff spaces, and let A be a compact nonempty subset of X. Let $f : A \to Y$ be continuous. Let $p : X \amalg Y \to X \amalg_f Y$ be the quotient map.

a) Show that p is a closed map.

b) Show that $p^{-1}(z)$ is a compact nonempty subset of $X \amalg Y$ for all $z \in X \amalg_f Y$.

<u>Definition 2.3.</u> The map $\Phi : X \to X \amalg_f Y$, which is the composite $X \hookrightarrow X \amalg Y \to X \amalg_f Y$, is called the <u>characteristic map.</u>

Theorem 2.4. Let X and Y be Hausdorff spaces, and let A be a compact subset of X. Let $f: A \to Y$ be continuous. Then $X \amalg_f Y$ is Hausdorff.

proof. Let $z_1, z_2 \in X \amalg_f Y$, $z_1 \neq z_2$. Then $p^{-1}(z_1) \cap p^{-1}(z_2) = \emptyset$, where $p: X \amalg Y \to X \amalg_f Y$ is the quotient map. Ex 2.2 b $\Rightarrow$ $p^{-1}(z_1)$ and $p^{-1}(z_2)$ are compact. Since $X \amalg Y$ is Hausdorff, $p^{-1}(z_1)$ and $p^{-1}(z_2)$ have open neighborhoods U_1 and U_2, respectively, with $U_1 \cap U_2 = \emptyset$. Ex 2.2 a $\Rightarrow$ p is a closed map. Lemma 1.2 $\Rightarrow$ for $i = 1,2$, z_i has a neighborhood V_i with $p^{-1}(V_i) \subset U_i$. Since $U_1 \cap U_2 = \emptyset$, it follows that $V_1 \cap V_2 = \emptyset$. □

Recall the following from the fall semester: Let X and Y be topological spaces and let $h: X \to Y$ be a function. Then $\ker h$ is an equivalence relation on X defined by $x \sim x'$ if $h(x) = h(x')$. The corresponding quotient space is denoted by $X/\ker h$. There exists an injection $\varphi: X/\ker h \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "p"'] & & Y \\ & X/\ker h \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Proposition 2.5. Let X and Y be topological spaces, and let $h: X \to Y$ be an identification (i.e., $h^{-1}(U)$ is open in X if and only if U is open in Y).
and h is a surjection
Then the map

$$\varphi: X/\ker h \to Y, \quad [x] \mapsto h(x),$$

is a homeomorphism.

proof. See the lecture notes of "Introduction to Algebraic Topology", fall 2015, Corollary 3.10. □

Lemma 2.6. Let X and Y be compact Hausdorff spaces, and let A be a closed subset of X. Let $f: A \to Y$ be continuous. Let W be a compact Hausdorff space. Assume there is a continuous surjection $h: X \amalg Y \to W$ with the following property: For $u, v \in X \amalg Y$, $u \sim v$ if and only if $h(u) = h(v)$. Then the map

Equiv. rel. defined by f.

$$\varphi : X \amalg_f Y \to W, \quad [u] \mapsto h(u),$$

is a homeomorphism.

proof. Let $p: X \amalg Y \to X \amalg_f Y$ be the quotient map. Consider the diagram

$$\begin{tikzcd} X \amalg Y \arrow[rr, "h"] \arrow[dr, "p"'] & & W \\ & X \amalg_f Y \arrow[ur, "\varphi"'] & \end{tikzcd} \qquad \varphi : [u] \mapsto h(u)$$

The property of h implies that $X \amalg_f Y$ equals $(X \amalg Y)/\ker h$. Then the diagram commutes. Since $X \amalg Y$ is compact and since W is Hausdorff, it follows that h is a closed map. Thus h is an identification. Proposition 2.5 $\Rightarrow$ φ is a homeomorphism. $\square$

Let $D^n = \{ x \in \mathbb{R}^n \mid \|x\| \le 1 \}$

and $S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$.

Then $D^n - S^{n-1}$ is the ~~closed~~ open unit disk with center 0.

Definition 2.7. An *n-cell* e^n (sometimes denoted by e) is a homeomorphic copy of $D^n - S^{n-1}$.

Here: $\left. \begin{array}{l} D^0 = \{0\} \\ S^{-1} = \emptyset \end{array} \right\}$ e^0 is a point

Definition 2.8. Let Y be a Hausdorff space and let $\phi : S^{n-1} \to Y$ be continuous. We say that the space

$$Y_\phi = D^n \amalg_\phi Y$$

is obtained from Y by attaching an n-cell via ϕ.

Let

$$\Phi : D^n \hookrightarrow D^n \amalg Y \to D^n \amalg_\phi Y = Y_\phi$$

be the characteristic map. Identify Y with its image in Y_ϕ via $y \mapsto [y]$. Then

1) Y_ϕ is Hausdorff (Thm 2.4).

2) Y_ϕ is compact if Y is compact.

3) Φ can be considered as a function of pairs $(D^n, S^{n-1}) \to (Y_\phi, Y)$. Thus $\Phi | S^{n-1}$ can be considered as the attaching map ϕ.

4) $\Phi(D^n - S^{n-1})$ is an n-cell, which is an open subset of Y_ϕ.

Definition 2.9. A continuous map

$$g : (X, A) \to (Y, B)$$

is called a relative homeomorphism, if the restriction $g|(X-A) : X - A \to Y - B$ is a homeomorphism.

Example 2.10.

1) The characteristic map $\Phi : (D^n, S^{n-1}) \to (Y_\phi, Y)$ is a relative homeomorphism.

2) Let X be a topological space, and let U and A be subspaces of X. Assume $\bar{U} \subset A^\circ$, where A° denotes the interior of A. Then the inclusion

$$i : (X-U, A-U) \hookrightarrow (X, A)$$

is a relative homeomorphism. Notice that i is just the excision map inducing isomorphisms

$$i_* : H_n(X-U, A-U) \to H_n(X, A), \quad \text{for all } n.$$

3) Let X be a compact Hausdorff space, and let A be a closed subset of X. Let $p : (X, A) \to (X/A, p(A))$ be the quotient map. Then p is a relative homeomorphism.

<u>Theorem 2.11.</u> Let Z be a compact Hausdorff space, let Y be a closed subset of Z, and let e be an n-cell in Z with $e \cap Y = \phi$. Assume there is a relative homeomorphism

$$\Phi : (D^n, S^{n-1}) \to (e \cup Y, Y).$$

Let $f = \phi| : S^{n-1} \to Y$. Then the map

$$g : Y_f = D^n \amalg_f Y \to e \cup Y, \quad [u] \mapsto (\Phi \amalg id_Y)(u),$$

is a homeomorphism.

<u>proof.</u> We will use Lemma 2.6 to prove the claim. Let

$$h = \Phi \amalg id_Y : D^n \amalg Y \to e \cup Y.$$

Then h is a continuous surjection. Thus $D^n \amalg Y$ compact $\Rightarrow e \cup Y = h(D^n \amalg Y)$ is compact. We must show that h satisfies the condition of Lemma 2.6.

Thus, we must show that for $u, v \in D^n \amalg Y$, $u \sim v$ if and only if $h(u) = h(v)$. Assume $u \sim v$. We may assume that $u \in S^{n-1}$. Then

1) $v \in Y$ and $v = \phi(u)$, or

2) $v \in S^{n-1}$ and $\phi(v) = \phi(u)$.

In both cases, $h(u) = h(v)$. Assume then that $h(u) = h(v)$. Then, either $u, v \in D^n$, $u, v \in Y$, or $u \in D^n$ and $v \in Y$ (similarly, $v \in D^n$ and $u \in Y$). If $u, v \in Y$, then $u = h(u) = h(v) = v$. Hence $u \sim v$. If $u \in D^n$ and $v \in Y$, then $\phi(u) = h(u) = h(v) = v$. Hence $u \sim v$. Assume $u, v \in D^n$. Since, by assumption, the restriction $\phi|D^n - S^{n-1} : D^n - S^{n-1} \to (e \cup Y) - Y = e$ is a homeomorphism, it follows that $u, v \in S^{n-1}$ and $\phi(u) = \phi(u) = \phi(v) = \phi(v)$. Thus $u \sim v$. It now follows from Lemma 2.6, that the map

$$g : Y_\phi \to e \cup Y, \quad [u] \mapsto (\phi \amalg id_Y)(u) = h(u),$$

is a homeomorphism. $\square$

The relative homeomorphism

$$\phi : (D^n, S^{n-1}) \to (e \cup Y, Y)$$

is also called a characteristic map.

3. Homology and Attaching Cells

Definition 3.1. Let X be a Hausdorff space. We call X locally compact, if for every $x \in X$ and for every open neighborhood U of x, there is an open subset W of X such that $x \in W \subset \overline{W} \subset U$ and $\overline{W}$ is compact.