

Exercise 6-7

2016-02-24

- 6 a) d : nature death rate of prey;
 S : nature death rate of predator;
 x_i : attack rate of i -th predator;
 γ : conversion factor of attacked prey into predator offspring

The following is one possible mechanism about the term $aN^2(1-\frac{N}{K})$ in the prey dynamic:



where a represents birth rate of prey, $1-\frac{2N}{K}$ ($=1-\frac{N}{K}$) is the proportion of empty sites. Here we can consider offspring will survival when they settle down on the unoccupied site ^{only}.

b) Prey dynamic without predators:

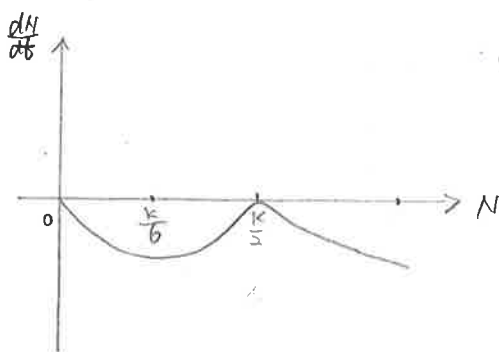
$$\frac{dN}{dt} = aN^2(1-\frac{N}{K}) - dN := f(N)$$

$$\begin{aligned} \Rightarrow f(N) &= \frac{a}{K}N(-N^2 + KN - \frac{dK}{a}) \\ &= -\frac{a}{K}N(N^2 - KN + \frac{dK}{a}) \end{aligned}$$

$$\text{As } f(N) = 0 \Rightarrow N_1 = 0$$

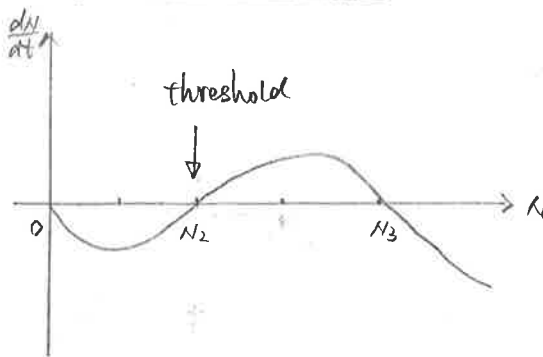
$$\begin{cases} N_{2,3} = \frac{K \pm \sqrt{K^2 - 4\frac{dK}{a}}}{2} & \text{if } K > \frac{4d}{a} \\ N_2 = N_3 = \frac{K}{2} & \text{if } K = \frac{4d}{a} \end{cases}$$

Case I: $K = \frac{4d}{a}$



Not Allee Effect

Case II: $K > \frac{4d}{a}$



Strong Allee Effect

In ecology, survival and reproduction of population are often limited when population size is small. This may lead to a population decline at low population densities. Further, the growth rate of population will become zero or negative value. This phenomenon is so-called Allee effect which indicates that there is a minimal population size necessary for a population to maintain itself. And Allee effect may arise from different mechanisms. For instance, the distribution of population in the region is so decentralized that they are difficult to finding mates. Or predator abundance lead to difficulty in survival of prey population.

c). Prey-predator dynamic with only a single predator type :

$$\textcircled{*} \begin{cases} \frac{1}{N} \dot{N} = aN \left(1 - \frac{N}{K}\right) - d - xP \\ \frac{1}{P} \dot{P} = \gamma xN - \delta \end{cases}$$

Positive equilibrium

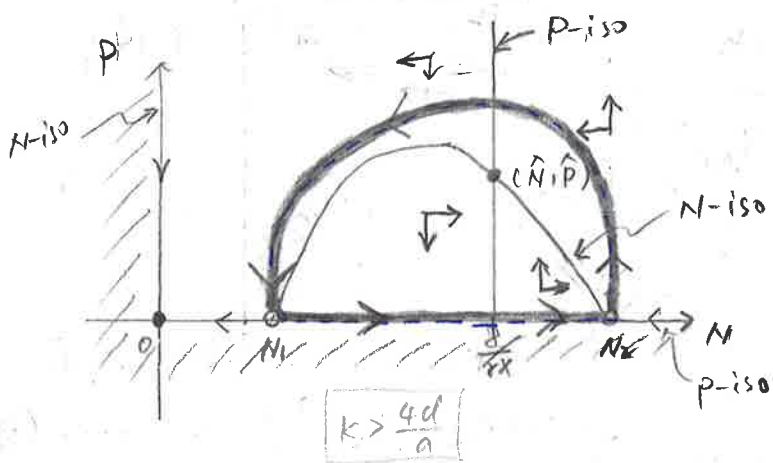
$$\begin{cases} \frac{\dot{P}}{P} = 0 \Rightarrow \hat{N}(x) = \frac{\delta}{\gamma x} \\ \frac{\dot{N}}{N} = 0 \Rightarrow \hat{P}(x) = \frac{a\gamma x}{x} \hat{N}(x) \left(1 - \frac{\hat{N}(x)}{K}\right) - \frac{d}{x} \end{cases}$$

Trivial equilibrium :

$$(0, 0)$$

boundary equilibrium :

$$(N_1, 0), (N_2, 0)$$



When x increases, the P -iso ($N = \frac{\delta}{\gamma x}$) will move to region $[0, N_1]$. Following the arrows, all orbits will converge to the trivial equilibrium $(0, 0)$. That is, prey and predator will extinct when x is very high.

From the Phase-plane analysis, you can find the flow like above figure. N_1 and N_2 are two saddle nodes. And there exists a heteroclinic cycle which connects N_1 and N_2 . Thus, the following we discuss the possibility of cycles in two cases.

Case I: Inside of heteroclinic cycle.

The stability of positive equilibrium (\hat{N}, \hat{P}) depends on the eigenvalues of Jacobian matrix of equation $(*)$ at values (\hat{N}, \hat{P}) . Further, from the Poincaré-Bendixon Theorem, we know that there exists a limit cycle or an equilibrium in the bounded heteroclinic cycle. Thus limit cycle is possible in the inside of heteroclinic cycle.

Case II: outside of heteroclinic cycle.

Following the arrow, all orbits which start from the region of outside of heteroclinic cycle will go to $(0,0)$ point.

d). Rewrite the prey-predator as the standard form.

$$\frac{\dot{n}_i}{n_i} = \gamma x_i E - \delta, \quad i = 1, 2, \dots,$$

$$\dot{E} = E \left(a E \left(1 - \frac{E}{K} \right) - d - \sum_j x_j n_j \right)$$

Invasor dynamic:

$$\frac{\dot{m}}{m} = \gamma \gamma E - \delta$$

Invasion fitness:

$$S_E(\gamma) = \gamma \langle \gamma E(t) \rangle - \delta. \quad \Rightarrow \text{Essential dimension of the environment is one.}$$

From the principle of selective neutrality,

$$S_E(x) = 0 \Rightarrow \langle \bar{E}(x) \rangle = \frac{\beta}{\gamma}$$

Thus the explicit expression of invasion fitness is

$$S_x(y) = \beta \left(\frac{y}{x} - 1 \right).$$

e). $S_x(y) > 0 \Rightarrow y > x$, that is, evolution will maximize the predator's attack rate. However, when x becomes very values, both the prey and the predator will catastrophic extinct.

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a). Rewrite the prey-predator dynamic as the standard form:

$$\frac{\dot{n}_i}{n_i} = \gamma E - \delta, \quad i=1,2,\dots$$

$$E = \frac{x_i N(x)}{1 + h x_i N(x)}, \quad x = (x_1, x_2, \dots)$$

Invader dynamic:

$$\frac{\dot{m}}{m} = \gamma E(y,t) - \delta.$$

Invasion fitness:

$$S_{E(y)}(y) = \gamma \langle \bar{E}(y,t) \rangle - \delta.$$

Now $\langle \bar{E}(y,t) \rangle$ is not a constant, but is a function of y .

Therefore, the environment is infinite-dimensional.

b). From the principle of selective neutrality of resident,

$$S_{E(x_i)}(x_i) = \gamma \langle \bar{E}(x_i,t) \rangle - \delta = 0 \quad \forall i,$$

$$\Rightarrow \langle \bar{E}(x_i,t) \rangle = \left\langle \frac{x_i N(x)}{1 + h x_i N(x)} \right\rangle = \frac{\beta}{\gamma}, \quad \forall i.$$

In addition, then selection gradient for monomorphic resident,

$$\left[\partial_y S_E(y) \right]_{y=x} = \left\langle \frac{\gamma N(x)}{(1 + h y N(x))^2} \right\rangle \Big|_{y=x} = \left\langle \frac{\gamma N(x)}{(1 + h x N(x))^2} \right\rangle > 0,$$

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which implies that only the invader strategy $y > x$ can invade successfully for any given resident strategy x .

In monomorphic resident, for any $x_j > x_i$,

$$S_{x_i}(x_j) > 0 \Rightarrow \left\langle \frac{x_j N(x_i)}{1+h x_j N(x_i)} \right\rangle > \frac{d}{r}$$

$$S_{x_j}(x_i) = 0 \Rightarrow \left\langle \frac{x_i N(x_i)}{1+h x_i N(x_i)} \right\rangle = \frac{d}{r}$$

$$\Rightarrow \left\langle \frac{x_i N(x_i)}{1+h x_i N(x_i)} \right\rangle < \left\langle \frac{x_j N(x_i)}{1+h x_j N(x_i)} \right\rangle$$

In fact, for any $X = (x_1, \dots)$, we always have

$$\left\langle \frac{x_i N(X)}{1+h x_i N(X)} \right\rangle < \left\langle \frac{x_j N(X)}{1+h x_j N(X)} \right\rangle \text{ whenever } x_j > x_i.$$

It follows that coexistence is not possible. (If coexistence is possible, then $\left\langle \frac{x_i N(X)}{1+h x_i N(X)} \right\rangle$ should equal to $\left\langle \frac{x_j N(X)}{1+h x_j N(X)} \right\rangle$).

c). Since coexistence is not possible and the selection gradient $\left[\partial_y S_E(y) \right]_{y=x} > 0$, then evolution will increase the predator's attack rate x . Return the original prey-predator with only one type predator,

$$\frac{\dot{N}}{N} = a N \left(1 - \frac{N}{K}\right) - d - \frac{xP}{1+h x N}$$

$$\frac{\dot{P}}{P} = \frac{r x N}{1+h x N} - \delta,$$

if $r > \delta h$.

equilibrium: $(0, 0)$, $(\hat{N}, \hat{P}) = \left(\frac{d}{r(\delta - \delta h)}, \frac{1}{\delta} \hat{N} \left(a \hat{N} \left(1 - \frac{\hat{N}}{K}\right) - d \right) \right)$.

Through the same argument of Exercise 6 (c), we can find that the evolution of x will lead to the catastrophic extinction of both prey and predator population.