Settlers and Floaters

(Case: settling as an irreversible unimolecular process and resource harvesting with Holling type I functional response)

Constant environment

RESIDENT DYNAMICS

$$\frac{dR}{dt} = \alpha\,(R_{in} - R) - R \sum_{j=1}^{k} (\beta_1 N_{1,j} + \beta_2 N_{2,j});\ (*\ \text{resource}\ *)$$

$$\frac{dN_{1,i}}{dt} = R\,(\gamma_1 \beta_1 N_{1,i} + \gamma_2 \beta_2 N_{2,i}) - (\delta_1 + \alpha + x_i)\,N_{1,i};\ (*\ \text{floaters}\ *)$$

$$\frac{dN_{2,i}}{dt} = x_i N_{1,i} \left(1 - \frac{1}{K} \sum_{j=1}^{k} N_{2,j}\right) - \delta_2 N_{2,i};\ (*\ \text{settlers}\ *)$$

WRITE AS

$$\frac{dR}{dt} = \alpha\,(R_{in} - R) - R \sum_{j=1}^{k} (\beta_1 N_{1,j} + \beta_2 N_{2,j});\ (*\ \text{resource}\ *)$$

$$\frac{d}{dt}\begin{pmatrix} N_{1,i} \\ N_{2,i} \end{pmatrix} = \begin{pmatrix} \gamma_1 \beta_1 \boxed{R} - \delta_1 - \alpha - x_i, & \gamma_2 \beta_2 \boxed{R} \\ x_i \boxed{\left(1 - \frac{1}{K} \sum_{j=1}^{k} N_{2,j}\right)}, & -\delta_2 \end{pmatrix} \begin{pmatrix} N_{1,i} \\ N_{2,i} \end{pmatrix}$$

INVADER DYNAMICS

$$\frac{d}{dt}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 \beta_1 \boxed{R} - \delta_1 - \alpha - y, & \gamma_2 \beta_2 \boxed{R} \\ y \boxed{\left(1 - \frac{1}{K} \sum_{j=1}^{k} N_{2,j}\right)}, & -\delta_2 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

Invasion in the virgin environment

ENVIRONMENT

$N_1 = N_2 = 0$

$$\frac{dR}{dt} = \alpha\,(R_{in} - R) \implies R \longrightarrow R_{in}$$

INVADER DYNAMICS

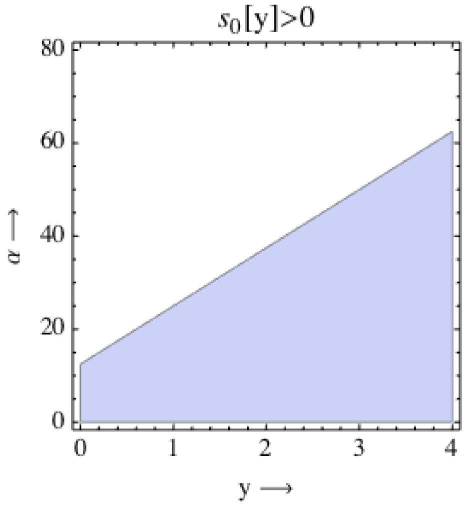
$$\frac{d}{dt}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 \beta_1 \boxed{R_{in}} - \delta_1 - \alpha - y), & \gamma_2 \beta_2 \boxed{R_{in}} \\ y \boxed{1}, & -\delta_2 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

$< ------------ A_0[y] ---- ---- >$

```
Clear[α, β1, β2, γ1, γ2, δ1, δ2, Rin, K];
A0[y_] := {{γ1 β1 Rin - δ1 - α - y, γ2 β2 Rin}, {y, -δ2}};
Eigenvalues[A0[y]] // Simplify
```

$$\left\{\frac{1}{2}\left(-y-\alpha+\text{Rin}\,\beta1\,\gamma1-\delta1-\delta2-\sqrt{(y+\alpha-\text{Rin}\,\beta1\,\gamma1+\delta1+\delta2)^2-4\,((y+\alpha+\delta1)\,\delta2-\text{Rin}\,(y\,\beta2\,\gamma2+\beta1\,\gamma1\,\delta2))}\right),\right.$$
$$\left.\frac{1}{2}\left(-y-\alpha+\text{Rin}\,\beta1\,\gamma1-\delta1-\delta2+\sqrt{(y+\alpha-\text{Rin}\,\beta1\,\gamma1+\delta1+\delta2)^2-4\,((y+\alpha+\delta1)\,\delta2-\text{Rin}\,(y\,\beta2\,\gamma2+\beta1\,\gamma1\,\delta2))}\right)\right\}$$

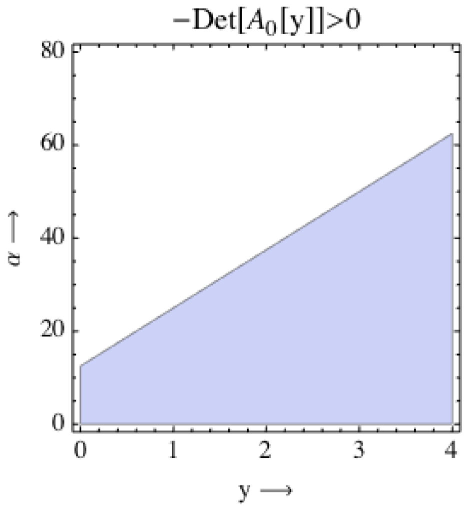
```
(* Invasion condition for virgine environment svir[y] *)

s0[y_] := 1/2 (-y - α + Rin β1 γ1 - δ1 - δ2 + Sqrt[(y + α - Rin β1 γ1 + δ1 + δ2)^2 - 4 ((y + α + δ1) δ2 - Rin (y β2 γ2 + β1 γ1 δ2))]);

α = 1.; β1 = 1.; β2 = 1.; γ1 = .9; γ2 = .9; δ1 = 1.; δ2 = 1.; Rin = 15.; K = 10.;

RegionPlot[s0[y] > 0, {y, 0, 4}, {α, 0, 80}, PlotLabel → "s₀[y]>0", FrameLabel → {"y ⟶", "α ⟶"}, ImageSize → Small]

Clear[α, β1, β2, γ1, γ2, δ1, δ2, Rin, K];
```




```
(* Using -Det[A₀[y]] as fitness proxy *)
Det[A0[y]]
```

-Rin y β2 γ2 + y δ2 + α δ2 - Rin β1 γ1 δ2 + δ1 δ2

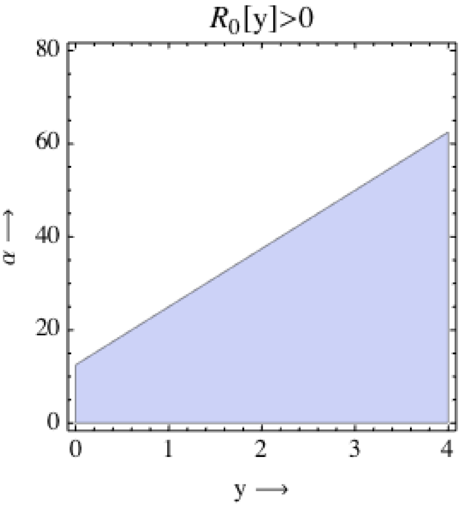
```
Det0[y_] := -Rin y β2 γ2 + y δ2 + α δ2 - Rin β1 γ1 δ2 + δ1 δ2;

α = 1.; β1 = 1.; β2 = 1.; γ1 = .9; γ2 = .9; δ1 = 1.; δ2 = 1.; Rin = 15.; K = 10.;

RegionPlot[Det0[y] < 0, {y, 0, 4}, {α, 0, 80},
 PlotLabel → "-Det[A₀[y]]>0", FrameLabel → {"y ⟶", "α ⟶"}, ImageSize → Small]

Clear[α, β1, β2, γ1, γ2, δ1, δ2, Rin, K];
```

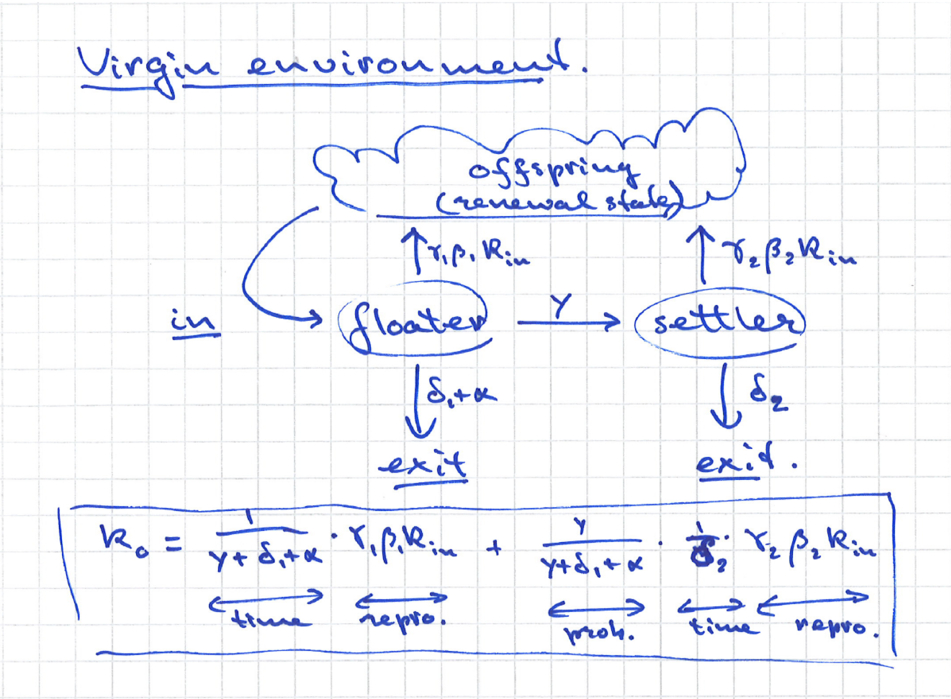


```
(* Using R₀[y]-1 as fitness proxy *)
```

$$\text{R0[y\_] := } \frac{1}{y + \alpha + \delta 1} \gamma 1\, \beta 1\, \text{Rin} + \frac{y}{y + \alpha + \delta 1} \frac{1}{\delta 2} \gamma 2\, \beta 2\, \text{Rin};$$

```
α = 1.; β1 = 1.; β2 = 1.; γ1 = .9; γ2 = .9; δ1 = 1.; δ2 = 1.; Rin = 15.; K = 10.;

RegionPlot[R0[y] > 1, {y, 0, 4}, {α, 0, 80}, PlotLabel → "R₀[y]>0", FrameLabel → {"y ⟶", "α ⟶"}, ImageSize → Small]

Clear[α, β1, β2, γ1, γ2, δ1, δ2, Rin, K];
```






```
(* Viability threshold *)

Solve[Det0[x] == 0, x]
```

$$\left\{\left\{x \to \frac{\alpha\,\delta 2 - \text{Rin}\,\beta 1\,\gamma 1\,\delta 2 + \delta 1\,\delta 2}{\text{Rin}\,\beta 2\,\gamma 2 - \delta 2}\right\}\right\}$$

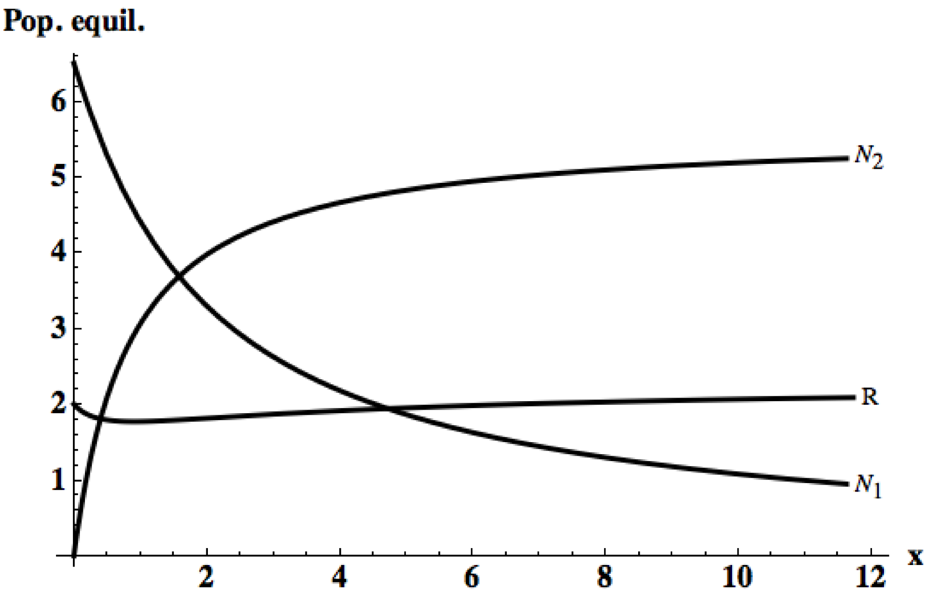
$$\text{xCrit := Max}\left[0, \frac{\alpha\,\delta 2 - \text{Rin}\,\beta 1\,\gamma 1\,\delta 2 + \delta 1\,\delta 2}{\text{Rin}\,\beta 2\,\gamma 2 - \delta 2}\right];$$

Monomorphic resident population

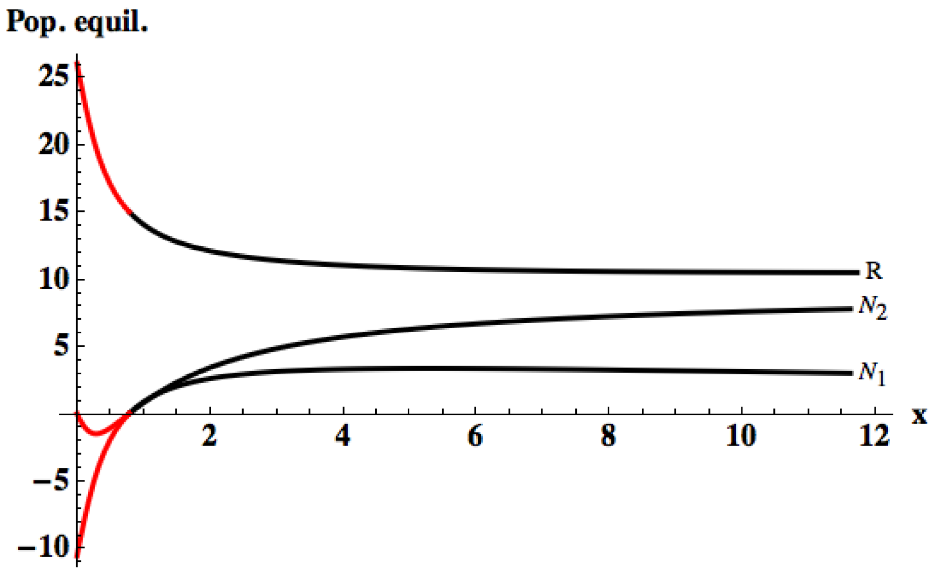
■ Equilibrium

```
eqs =
  {0 == α (Rin - R) - R (β1 N1 + β2 N2),
   0 == R (γ1 β1 N1 + γ2 β2 N2) - (δ1 + α + x) N1,
   0 == x N1 (1 - N2 / K) - δ2 N2};

vrs = {R, N1, N2};

Solve[eqs, vrs] // Simplify;

{

  (* first solution: *)
  {N2 → 0, R → Rin, N1 → 0},

  (* second solution: *){N2 → ((-x^2 α - x α^2 - K x^2 β2 - K x α β2 + Rin x α β1 γ1 - x α δ1 - K x β2 δ1 +
          K x β1 δ2 + K α β1 δ2 + K β1 δ1 δ2 + √(-4 K x α β1 (x + α + δ1) ((x + α + δ1) δ2 - Rin (x β2 γ2 + β1 γ1 δ2)) +
              (x^2 (α + K β2) + K β1 (α + δ1) δ2 + x (α^2 + α (K β2 - Rin β1 γ1 + δ1) + K (β2 δ1 + β1 δ2)))^2))
        (x^2 α + x α^2 + K x^2 β2 + K x α β2 - Rin x α β1 γ1 + x α δ1 + K x β2 δ1 + K x β1 δ2 + K α β1 δ2 + K β1 δ1 δ2 +
          √(-4 K x α β1 (x + α + δ1) ((x + α + δ1) δ2 - Rin (x β2 γ2 + β1 γ1 δ2)) + (x^2 (α + K β2) + K β1 (α + δ1) δ2 +
                x (α^2 + α (K β2 - Rin β1 γ1 + δ1) + K (β2 δ1 + β1 δ2)))^2))) / (4 x β1 β2 (x + α + δ1) (Rin x α γ2 + K (x + α + δ1) δ2)),

   R → (x^2 α γ2 + x α^2 γ2 + K x^2 β2 γ2 + K x α β2 γ2 + Rin x α β1 γ1 γ2 + x α γ2 δ1 +
        K x β2 γ2 δ1 + 2 K x β1 γ1 δ2 + 2 K α β1 γ1 δ2 - K x β1 γ2 δ2 - K α β1 γ2 δ2 + 2 K β1 γ1 δ1 δ2 - K β1 γ2 δ1 δ2 +
        γ2 √(-4 K x α β1 (x + α + δ1) ((x + α + δ1) δ2 - Rin (x β2 γ2 + β1 γ1 δ2)) + (x^2 (α + K β2) + K β1 (α + δ1) δ2 + x
              (α^2 + α (K β2 - Rin β1 γ1 + δ1) + K (β2 δ1 + β1 δ2)))^2)) / (2 β1 (x (α γ1 + K β2 (γ1 - γ2)) γ2 + K β1 γ1 (γ1 - γ2) δ2)),

   N1 → - 1/(2 x β1 (x + α + δ1)) (x^2 α + x α^2 + K x^2 β2 + K x α β2 - Rin x α β1 γ1 + x α δ1 + K x β2 δ1 +
        K x β1 δ2 + K α β1 δ2 + K β1 δ1 δ2 + √(-4 K x α β1 (x + α + δ1) ((x + α + δ1) δ2 - Rin (x β2 γ2 + β1 γ1 δ2)) +
            (x^2 (α + K β2) + K β1 (α + δ1) δ2 + x (α^2 + α (K β2 - Rin β1 γ1 + δ1) + K (β2 δ1 + β1 δ2)))^2))},

  (* third solution: THIS IS THE ONE WE NEED *)
  {N2 → -((x^2 α + x α^2 + K x^2 β2 + K x α β2 - Rin x α β1 γ1 + x α δ1 + K x β2 δ1 + K x β1 δ2 +
          K α β1 δ2 + K β1 δ1 δ2 - √(-4 K x α β1 (x + α + δ1) ((x + α + δ1) δ2 - Rin (x β2 γ2 + β1 γ1 δ2)) +
              (x^2 (α + K β2) + K β1 (α + δ1) δ2 + x (α^2 + α (K β2 - Rin β1 γ1 + δ1) + K (β2 δ1 + β1 δ2)))^2))
        (x^2 α + x α^2 + K x^2 β2 + K x α β2 - Rin x α β1 γ1 + x α δ1 + K x β2 δ1 - K x β1 δ2 - K α β1 δ2 - K β1 δ1 δ2 +
          √(-4 K x α β1 (x + α + δ1) ((x + α + δ1) δ2 - Rin (x β2 γ2 + β1 γ1 δ2)) + (x^2 (α + K β2) + K β1 (α + δ1) δ2 +
                x (α^2 + α (K β2 - Rin β1 γ1 + δ1) + K (β2 δ1 + β1 δ2)))^2))) / (4 x β1 β2 (x + α + δ1) (Rin x α γ2 + K (x + α + δ1) δ2)),

   R → (x^2 α γ2 + x α^2 γ2 + K x^2 β2 γ2 + K x α β2 γ2 + Rin x α β1 γ1 γ2 + x α γ2 δ1 +
        K x β2 γ2 δ1 + 2 K x β1 γ1 δ2 + 2 K α β1 γ1 δ2 - K x β1 γ2 δ2 - K α β1 γ2 δ2 + 2 K β1 γ1 δ1 δ2 - K β1 γ2 δ1 δ2 -
        γ2 √(-4 K x α β1 (x + α + δ1) ((x + α + δ1) δ2 - Rin (x β2 γ2 + β1 γ1 δ2)) + (x^2 (α + K β2) + K β1 (α + δ1) δ2 + x
              (α^2 + α (K β2 - Rin β1 γ1 + δ1) + K (β2 δ1 + β1 δ2)))^2)) / (2 β1 (x (α γ1 + K β2 (γ1 - γ2)) γ2 + K β1 γ1 (γ1 - γ2) δ2)),

   N1 → - 1/(2 x β1 (x + α + δ1)) (x^2 α + x α^2 + K x^2 β2 + K x α β2 - Rin x α β1 γ1 + x α δ1 + K x β2 δ1 +
        K x β1 δ2 + K α β1 δ2 + K β1 δ1 δ2 - √(-4 K x α β1 (x + α + δ1) ((x + α + δ1) δ2 - Rin (x β2 γ2 + β1 γ1 δ2)) +
            (x^2 (α + K β2) + K β1 (α + δ1) δ2 + x (α^2 + α (K β2 - Rin β1 γ1 + δ1) + K (β2 δ1 + β1 δ2)))^2))}

 };
```

```
(* Positive equilibrium monomorphic resident population *)

eqR[x_] := (x^2 α γ2 + x α^2 γ2 + K x^2 β2 γ2 + K x α β2 γ2 + Rin x α β1 γ1 γ2 + x α γ2 δ1 + K x β2 γ2 δ1 + 2 K x β1 γ1 δ2 + 2 K α β1 γ1 δ2 -
      K x β1 γ2 δ2 - K α β1 γ2 δ2 + 2 K β1 γ1 δ1 δ2 - K β1 γ2 δ1 δ2 - γ2 √(-4 K x α β1 (x + α + δ1) ((x + α + δ1) δ2 - Rin (x β2 γ2 + β1 γ1 δ2)) +
         (x^2 (α + K β2) + K β1 (α + δ1) δ2 + x (α^2 + α (K β2 - Rin β1 γ1 + δ1) + K (β2 δ1 + β1 δ2)))^2)) /
   (2 β1 (x (α γ1 + K β2 (γ1 - γ2)) γ2 + K β1 γ1 (γ1 - γ2) δ2));

eqN1[x_] := - 1/(2 x β1 (x + α + δ1)) (x^2 α + x α^2 + K x^2 β2 + K x α β2 - Rin x α β1 γ1 + x α δ1 + K x β2 δ1 +
      K x β1 δ2 + K α β1 δ2 + K β1 δ1 δ2 - √(-4 K x α β1 (x + α + δ1) ((x + α + δ1) δ2 - Rin (x β2 γ2 + β1 γ1 δ2)) +
         (x^2 (α + K β2) + K β1 (α + δ1) δ2 + x (α^2 + α (K β2 - Rin β1 γ1 + δ1) + K (β2 δ1 + β1 δ2)))^2));

eqN2[x_] := - ((x^2 α + x α^2 + K x^2 β2 + K x α β2 - Rin x α β1 γ1 + x α δ1 + K x β2 δ1 +
         K x β1 δ2 + K α β1 δ2 + K β1 δ1 δ2 - √(-4 K x α β1 (x + α + δ1) ((x + α + δ1) δ2 - Rin (x β2 γ2 + β1 γ1 δ2)) +
            (x^2 (α + K β2) + K β1 (α + δ1) δ2 + x (α^2 + α (K β2 - Rin β1 γ1 + δ1) + K (β2 δ1 + β1 δ2)))^2))
      (x^2 α + x α^2 + K x^2 β2 + K x α β2 - Rin x α β1 γ1 + x α δ1 + K x β2 δ1 - K x β1 δ2 - K α β1 δ2 - K β1 δ1 δ2 +
         √(-4 K x α β1 (x + α + δ1) ((x + α + δ1) δ2 - Rin (x β2 γ2 + β1 γ1 δ2)) + (x^2 (α + K β2) + K β1 (α + δ1) δ2 + x
               (α^2 + α (K β2 - Rin β1 γ1 + δ1) + K (β2 δ1 + β1 δ2)))^2))) / (4 x β1 β2 (x + α + δ1) (Rin x α γ2 + K (x + α + δ1) δ2));

(* Stability of the positive equilibrium *)

JacMat[x_] :=
  D[{α (Rin - R) - R (β1 N1 + β2 N2), R (γ1 β1 N1 + γ2 β2 N2) - (δ1 + α + x) N1, x N1 (1 - N2 / K) - δ2 N2}, {{R, N1, N2}}] /.
   {R → eqR[x], N1 → eqN1[x], N2 → eqN2[x]};

(* Parameter set 1 *)

α = 1; β1 = 1; β2 = 1; γ1 = 1; γ2 = 1; δ1 = 1; δ2 = 1; Rin = 15; K = 10;

Block[{xmin = 0, xmax = 12},
 Show[
  Plot[If[Max[Re[Eigenvalues[JacMat[x]]]] ≤ 0, {eqR[x], eqN1[x], eqN2[x]}],
   {x, xmin, xmax}, PlotStyle → {{Black, Thick}}],
  Plot[If[Max[Re[Eigenvalues[JacMat[x]]]] > 0, {eqR[x], eqN1[x], eqN2[x]}], {x, xmin, xmax}, PlotStyle → {{Red, Thick}}],
  Graphics[Text[" R ", {xmax, eqR[xmax]}, Background → White]],
  Graphics[Text[" N₁ ", {xmax, eqN1[xmax]}, Background → White]],
  Graphics[Text[" N₂ ", {xmax, eqN2[xmax]}, Background → White]],
  PlotRange → All, AxesOrigin → {0, 0}, AxesLabel → {"x", "Pop. equil."}, LabelStyle → Directive[Bold, 12]
 ]
]

Clear[α, β1, β2, γ1, γ2, δ1, δ2, Rin, K];
```



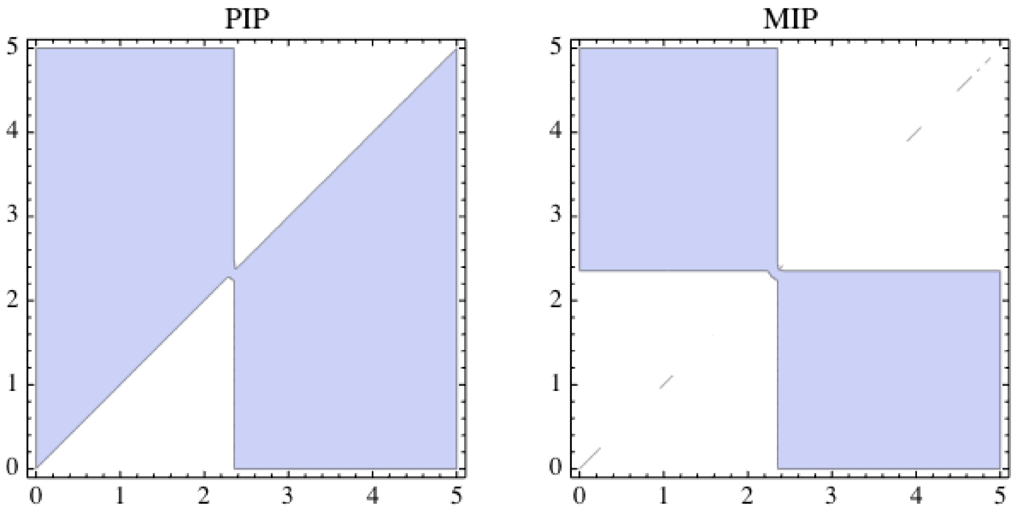
```
(* Parameter set 2 *)

α = 25; β1 = 1; β2 = 1; γ1 = 1; γ2 = 1; δ1 = 1; δ2 = 1; Rin = 15; K = 10;

Block[{xmin = 0, xmax = 12},
 Show[
  Plot[If[Max[Re[Eigenvalues[JacMat[x]]]] ≤ 0, {eqR[x], eqN1[x], eqN2[x]}],
   {x, xmin, xmax}, PlotStyle → {{Black, Thick}}],
  Plot[If[Max[Re[Eigenvalues[JacMat[x]]]] > 0, {eqR[x], eqN1[x], eqN2[x]}], {x, xmin, xmax}, PlotStyle → {{Red, Thick}}],
  Graphics[Text[" R ", {xmax, eqR[xmax]}, Background → White]],
  Graphics[Text[" N₁ ", {xmax, eqN1[xmax]}, Background → White]],
  Graphics[Text[" N₂ ", {xmax, eqN2[xmax]}, Background → White]],
  PlotRange → All, AxesOrigin → {0, 0}, AxesLabel → {"x", "Pop. equil."}, LabelStyle → Directive[Bold, 12]
 ]
]

Clear[α, β1, β2, γ1, γ2, δ1, δ2, Rin, K];
```



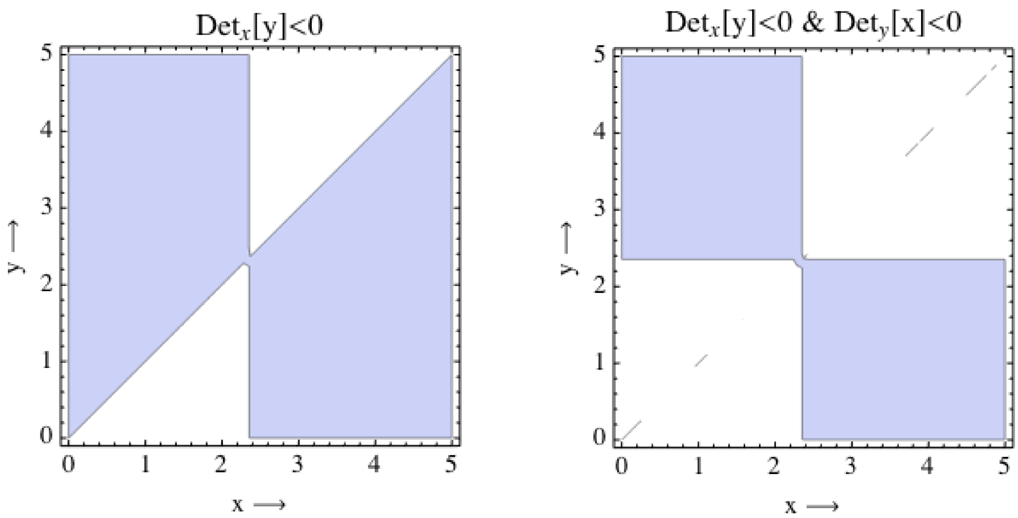
- **Invasion fitness and PIP**

```
(* Generator invader dynamics *)

A1[y_] := {{γ1 β1 R - δ1 - α - y, γ2 β2 R}, {y (1 - N2 / K), -δ2}};
```

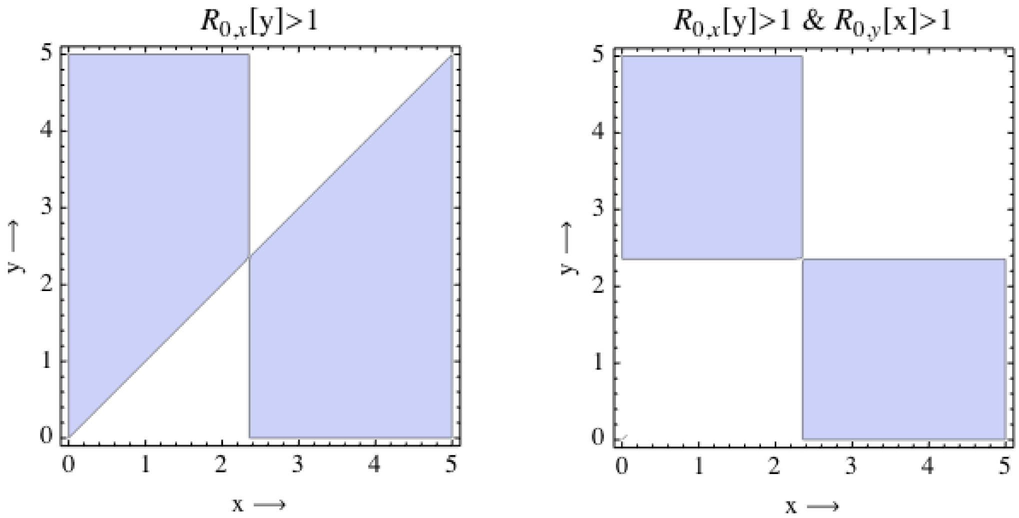
```
(* Invasion fitness *)

sx_[y_] :=
  Max[Eigenvalues[A1[y]]] /. {R → eqR[x], N1 → eqN1[x], N2 → eqN2[x]};

(* Example *)

α = 1.; β1 = 1.95; β2 = 1.; γ1 = .1; γ2 = .1; δ1 = 1.; δ2 = 1.; Rin = 15.; K = 10.;

PIP = RegionPlot[sx[y] > 0, {x, 0, 5}, {y, 0, 5}];

MIP = RegionPlot[sx[y] > 0 ∧ sy[x] > 0, {x, 0, 5}, {y, 0, 5}];

Row[{Show[PIP, PlotLabel → "PIP", ImageSize → Small], "     "
   Show[MIP, PlotLabel → "MIP", ImageSize → Small]}]

Clear[α, β1, β2, γ1, γ2, δ1, δ2, Rin, K];
```

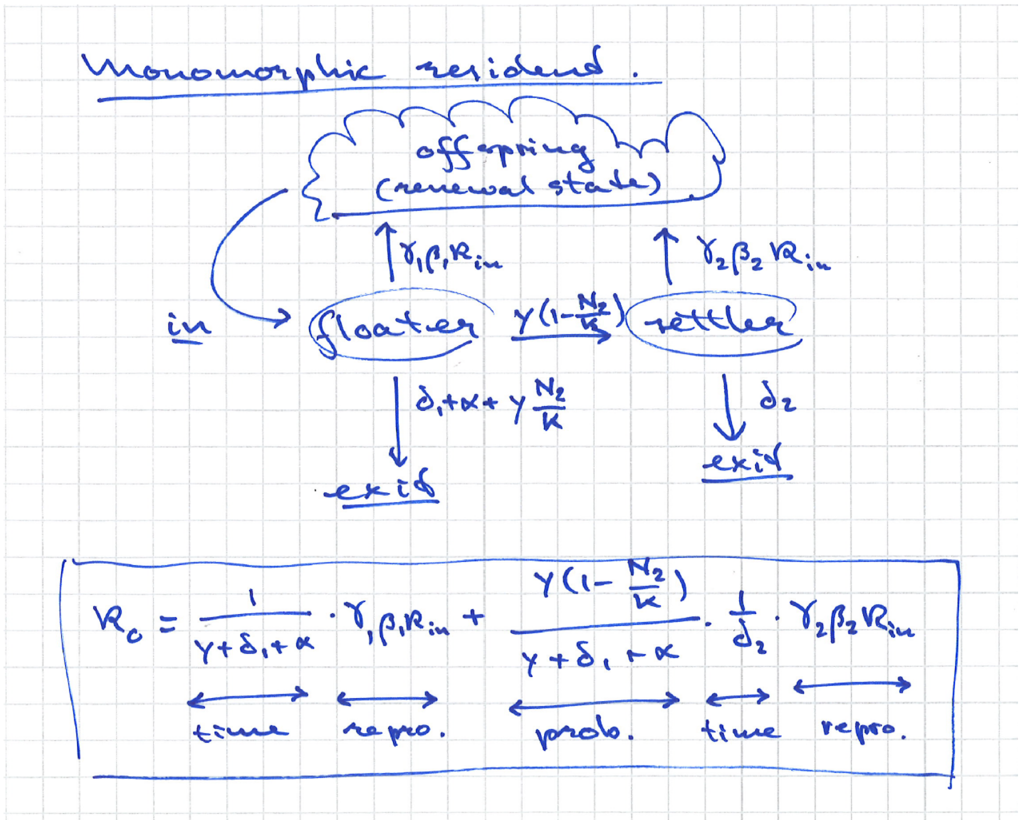



```
(* Fitness proxy: -Det[A1[y]] *)

Detx_[y_] := Det[A1[y]] /. {R → eqR[x], N1 → eqN1[x], N2 → eqN2[x]};

(* Example *)

α = 1.; β1 = 1.95; β2 = 1.; γ1 = .1; γ2 = .1; δ1 = 1.; δ2 = 1.; Rin = 15.; K = 10.;

Row[{
  RegionPlot[Detx[y] < 0, {x, xCrit, 5}, {y, xCrit, 5},
   PlotLabel → "Detx[y]<0", FrameLabel → {"x ⟶", "y ⟶"}, ImageSize → Small],
  "      ",
  RegionPlot[Detx[y] < 0 ∧ Dety[x] < 0, {x, xCrit, 5}, {y, xCrit, 5},
   PlotLabel → "Detx[y]<0 & Dety[x]<0", FrameLabel → {"x ⟶", "y ⟶"}, ImageSize → Small]
 }]

Clear[α, β1, β2, γ1, γ2, δ1, δ2, Rin, K];
```



```
(* Using R₀[y]-1 as fitness proxy ... *)

R0x_[y_] := 1/(y + α + δ1) γ1 β1 R + (y (1 - N2 / K))/(y + α + δ1) 1/δ2 γ2 β2 R /. {R → eqR[x], N1 → eqN1[x], N2 → eqN2[x]};

(* Example *)

α = 1.; β1 = 1.95; β2 = 1.; γ1 = .1; γ2 = .1; δ1 = 1.; δ2 = 1.; Rin = 15.; K = 10.;

Row[{
  RegionPlot[R0x[y] > 1, {x, xCrit, 5}, {y, xCrit, 5},
   PlotLabel → "R0,x[y]>1", FrameLabel → {"x ⟶", "y ⟶"}, ImageSize → Small],
  "      ",
  RegionPlot[R0x[y] > 1 ∧ R0y[x] > 1, {x, xCrit, 5}, {y, xCrit, 5},
   PlotLabel → "R0,x[y]>1 & R0,y[x]>1", FrameLabel → {"x ⟶", "y ⟶"}, ImageSize → Small]
 }]

Clear[α, β1, β2, γ1, γ2, δ1, δ2, Rin, K];
```





- **Why the vertical invasion boundary?**

```
Det[A1[y]] // Simplify
```

$$(y+\alpha+\delta 1)\ \delta 2 + R\left(\frac{(-K+N2)\ y\ \beta 2\ \gamma 2}{K} - \beta 1\ \gamma 1\ \delta 2\right)$$

Hence

$$\mathrm{Det}_x[y] = (y+\alpha+\delta 1)\ \delta 2 + R[x]\left(\frac{(-K+N2[x])\ y\ \beta 2\ \gamma 2}{K} - \beta 1\ \gamma 1\ \delta 2\right)$$

which is linear in y;

Solving $\mathrm{Det}_x[y] == 0$ for (x, y) always gives y = x as a solution;
Hence, $\mathrm{Det}_x[y] / (y - x)$ is independent of y, and so any solution (x, y) of
$0 == \mathrm{Det}_x[y] / (y - x)$ must be of the form x = constant;

```
Solve[0 == Det[A1[y]], y] /. {R → eqR[x], N1 → eqN1[x], N2 → eqN2[x]} // Simplify
```

$$\{\{y \to x\}\}$$

```
(* Det_x[y]/(y-x) is independent of y *)
Det[A1[y]] / (y - x) /. {R → eqR[x], N1 → eqN1[x], N2 → eqN2[x]} // Simplify
```

$$\Big(\delta 2\ \Big(x^2\ \alpha\ \gamma 1\ \gamma 2 - x\ \alpha^2\ \gamma 1\ \gamma 2 + K\ x^2\ \beta 2\ \gamma 1\ \gamma 2 - K\ x\ \alpha\ \beta 2\ \gamma 1\ \gamma 2 + \mathrm{Rin}\ x\ \alpha\ \beta 1\ \gamma 1^2\ \gamma 2 + 2\ K\ x\ \alpha\ \beta 2\ \gamma 2^2 - x\ \alpha\ \gamma 1\ \gamma 2\ \delta 1 - K\ x\ \beta 2\ \gamma 1\ \gamma 2\ \delta 1 + 2\ K\ x\ \beta 2\ \gamma 2^2\ \delta 1 + 2\ K\ x\ \beta 1\ \gamma 1^2\ \delta 2 - K\ x\ \beta 1\ \gamma 1\ \gamma 2\ \delta 2 + K\ \alpha\ \beta 1\ \gamma 1\ \gamma 2\ \delta 2 + K\ \beta 1\ \gamma 1\ \gamma 2\ \delta 1\ \delta 2 - \gamma 1\ \gamma 2\ \sqrt{\Big(-4\ K\ x\ \alpha\ \beta 1\ (x+\alpha+\delta 1)\ ((x+\alpha+\delta 1)\ \delta 2 - \mathrm{Rin}\ (x\ \beta 2\ \gamma 2 + \beta 1\ \gamma 1\ \delta 2)) + \big(x^2\ (\alpha + K\ \beta 2) + K\ \beta 1\ (\alpha+\delta 1)\ \delta 2 + x\ \big(\alpha^2 + \alpha\ (K\ \beta 2 - \mathrm{Rin}\ \beta 1\ \gamma 1 + \delta 1) + K\ (\beta 2\ \delta 1 + \beta 1\ \delta 2)\big)\big)^2\Big)}\Big)\Big) \Big/ (2\ x\ (x\ (\alpha\ \gamma 1 + K\ \beta 2\ (\gamma 1 - \gamma 2))\ \gamma 2 + K\ \beta 1\ \gamma 1\ (\gamma 1 - \gamma 2)\ \delta 2))$$

```
(* Numerator of Det[A1[y]]/(y-x) bexomes zero for which x ? *)
Solve[0 == x^2 α γ1 γ2 - x α^2 γ1 γ2 + K x^2 β2 γ1 γ2 - K x α β2 γ1 γ2 + Rin x α β1 γ1^2 γ2 +
   2 K x α β2 γ2^2 - x α γ1 γ2 δ1 - K x β2 γ1 γ2 δ1 + 2 K x β2 γ2^2 δ1 + 2 K x β1 γ1^2 δ2 - K x β1 γ1 γ2 δ2 +
   K α β1 γ1 γ2 δ2 + K β1 γ1 γ2 δ1 δ2 - γ1 γ2 Sqrt[(-4 K x α β1 (x + α + δ1) ((x + α + δ1) δ2 - Rin (x β2 γ2 + β1 γ1 δ2)) +
      (x^2 (α + K β2) + K β1 (α + δ1) δ2 + x (α^2 + α (K β2 - Rin β1 γ1 + δ1) + K (β2 δ1 + β1 δ2)))^2)], x] // Simplify
```

$$\left\{\{x \to 0\},\ \left\{x \to \frac{K\ \beta 1\ \gamma 1\ (-\gamma 1 + \gamma 2)\ \delta 2}{(\alpha\ \gamma 1 + K\ \beta 2\ (\gamma 1 - \gamma 2))\ \gamma 2}\right\},\ \left\{x \to -\frac{K\ \gamma 2\ (\alpha + \delta 1)\ (\alpha\ \beta 2\ \gamma 2 + \beta 2\ \gamma 2\ \delta 1 - \beta 1\ \gamma 1\ \delta 2)}{\gamma 1\ \left(\alpha^2\ \gamma 2 + \alpha\ \gamma 2\ (K\ \beta 2 - \mathrm{Rin}\ \beta 1\ \gamma 1 + \delta 1) + K\ (\beta 2\ \gamma 2\ \delta 1 - \beta 1\ \gamma 1\ \delta 2)\right)}\right\}\right\}$$

```
(* Denominator of ∂_yDet_x[y]/.{y→x} becomes zero for which x *)
Solve[0 == 2 x (x (α γ1 + K β2 (γ1 - γ2)) γ2 + K β1 γ1 (γ1 - γ2) δ2), x] // Simplify
```

$$\left\{\{x \to 0\},\ \left\{x \to -\frac{K\ \beta 1\ \gamma 1\ (\gamma 1 - \gamma 2)\ \delta 2}{(\alpha\ \gamma 1 + K\ \beta 2\ (\gamma 1 - \gamma 2))\ \gamma 2}\right\}\right\}$$

The numerator and denominator have a common root

$$x \to -\frac{K\ \beta_1\ \gamma_1\ (\gamma_1 - \gamma_2)\ \delta_2}{(\alpha\ \gamma_1 + K\ \beta_2\ (\gamma_1 - \gamma_2))\ \gamma_2};$$

This root is <u>positive</u> if and only if

$$1 < \frac{\gamma_2}{\gamma_1} < 1 + \frac{\alpha}{K\ \beta_2};$$

The left limit coincides with x = 0 and the right limit with x = +∞;

The numerator becomes zero for

$$x \to -\frac{K\ \gamma_2\ (\alpha + \delta_1)\ (\alpha\ \beta_2\ \gamma_2 + \beta_2\ \gamma_2\ \delta_1 - \beta_1\ \gamma_1\ \delta_2)}{\gamma_1\ \left(\alpha^2\ \gamma_2 + \alpha\ \gamma_2\ (K\ \beta_2 - R_{in}\ \beta_1\ \gamma_1 + \delta_1) + K\ (\beta_2\ \gamma_2\ \delta_1 - \beta_1\ \gamma_1\ \delta_2)\right)};$$

This root becomes zero for

$$\frac{\beta_1\ \gamma_1}{\alpha + \delta_1} == \frac{\beta_2\ \gamma_2}{\delta_2};$$

The root becomes ± infinity for

$$\frac{\beta_1\ \gamma_1}{\alpha + \delta_1} == \frac{\beta_2\ \gamma_2}{\delta_2} \cdot \frac{1 + \frac{\alpha}{K\ \beta_2}}{1 + \frac{\gamma_2}{\delta_2}\ \frac{\alpha}{K}\ R_{in}};$$

```
(* When does x* become infinity *)
```

$$\text{Solve}\left[\frac{\beta_1\,\gamma_1}{\alpha+\delta_1} == \frac{\beta_2\,\gamma_2}{\delta_2}\;\frac{1+\frac{\alpha}{K\,\beta_2}}{1+\frac{\gamma_2}{\delta_2}\,\frac{\alpha}{K}\,R_{in}},\ \gamma_2\right]\ /.\ \{\beta_1\to\beta1,\ \beta_2\to\beta2,\ \gamma_1\to\gamma1,\ \gamma_2\to\gamma2,\ \delta_1\to\delta1,\ \delta_2\to\delta2,\ R_{in}\to Rin\}\ //\ \text{Simplify}$$

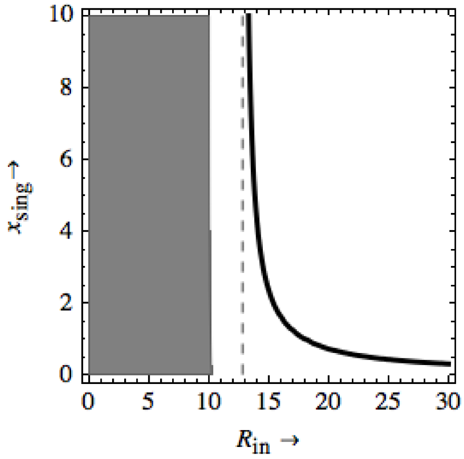
$$\left\{\left\{\gamma2\to\frac{K\,\beta1\,\gamma1\,\delta2}{\alpha^2+K\,\beta2\,\delta1+\alpha\,(K\,\beta2-Rin\,\beta1\,\gamma1+\delta1)}\right\}\right\}$$

Bifurcation plots

```
α = 1.; β1 = 1.95; β2 = 1.; γ1 = .1; γ2 = .1; δ1 = 1.; δ2 = 1.; Rin = 15.; K = 10.;

BP = ContourPlot[∂y Detx[y] /. {y → x}, {Rin, 0, 30},
   {x, 0, 10}, Contours → {0}, ContourShading → False, ContourStyle → {Thick}];
```

$$\text{XL = ContourPlot}\left[\boxed{Rin == \frac{(\alpha+K\,\beta2)\,\gamma2\,(\alpha+\delta1)-K\,\beta1\,\gamma1\,\delta2}{\alpha\,\beta1\,\gamma1\,\gamma2}},\ \{Rin, 0, 30\},\ \{x, 0, 10\},\ \text{ContourStyle}\to\{\text{Dashed}\}\right];$$

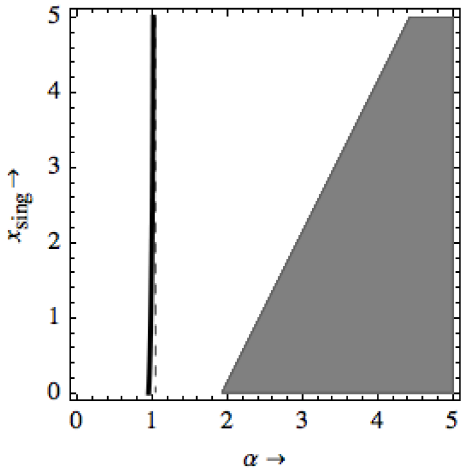
```
NV = RegionPlot[Det0[x] > 0, {Rin, 0, 30}, {x, 0, 10}, PlotStyle → Gray];

Show[{BP, XL, NV}, FrameLabel → {"Rin →", "xsing→"}, ImageSize → Small]

Clear[α, β1, β2, γ1, γ2, δ1, δ2, Rin, K];
```



```
α = 1.; β1 = 1.95; β2 = 1.; γ1 = .1; γ2 = .1; δ1 = 1.; δ2 = 1.; Rin = 15.; K = 10.;

BP = ContourPlot[∂y Detx[y] /. {y → x}, {α, 0, 5}, {x, 0, 5}, Contours → {0}, ContourShading → False, ContourStyle → {Thick}];
```

XL = ContourPlot[$\alpha == \frac{-K\,\beta2\,\gamma2 + Rin\,\beta1\,\gamma1\,\gamma2 - \gamma2\,\delta1 + \sqrt{\gamma2\,\left(-4\,K\,\beta2\,\gamma2\,\delta1 + \gamma2\,(K\,\beta2 - Rin\,\beta1\,\gamma1 + \delta1)^2 + 4\,K\,\beta1\,\gamma1\,\delta2\right)}}{2\,\gamma2}$,

```
    {α, 0, 5}, {x, 0, 5}, ContourStyle → {Dashed}];

NV = RegionPlot[Det0[x] > 0, {α, 0, 5}, {x, 0, 5}, PlotStyle → Gray];

Show[{BP, XL, NV}, FrameLabel → {"α →", "xsing→"}, ImageSize → Small]

Clear[α, β1, β2, γ1, γ2, δ1, δ2, Rin, K];
```

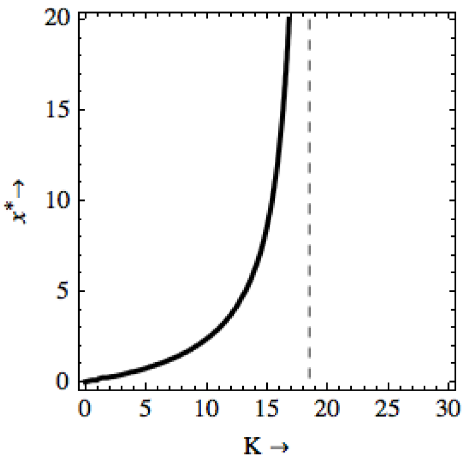


$$-\frac{\alpha\,\gamma2\,(\alpha - Rin\,\beta1\,\gamma1 + \delta1)}{\alpha\,\beta2\,\gamma2 + \beta2\,\gamma2\,\delta1 - \beta1\,\gamma1\,\delta2}$$

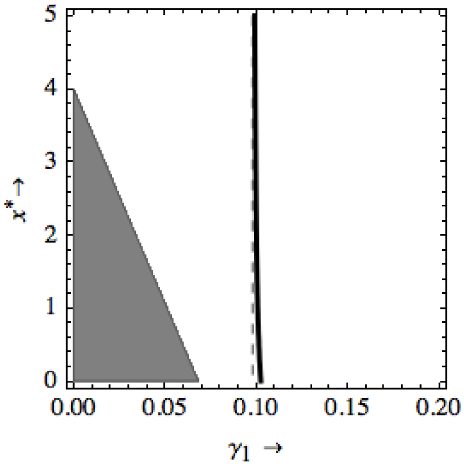
```
α = 1.; β1 = 1.95; β2 = 1.; γ1 = .1; γ2 = .1; δ1 = 1.; δ2 = 1.; Rin = 15.; K = 10.;

BP = ContourPlot[∂y Detx[y] /. {y → x}, {K, 0, 30},
    {x, 0, 20}, Contours → {0}, ContourShading → False, ContourStyle → {Thick}];
```

XL = ContourPlot[$K == -\frac{\alpha\,\gamma2\,(\alpha - Rin\,\beta1\,\gamma1 + \delta1)}{\alpha\,\beta2\,\gamma2 + \beta2\,\gamma2\,\delta1 - \beta1\,\gamma1\,\delta2}$, {K, 0, 30}, {x, 0, 20}, ContourStyle → {Dashed}];

```
NV = RegionPlot[Det0[x] > 0, {K, 0, 30}, {x, 0, 20}, PlotStyle → Gray];

Show[{BP, XL, NV}, FrameLabel → {"K →", "x*→"}, ImageSize → Small]

Clear[α, β1, β2, γ1, γ2, δ1, δ2, Rin, K];
```



```
α = 1.; β1 = 1.95; β2 = 1.; γ1 = .1; γ2 = .1; δ1 = 1.; δ2 = 1.; Rin = 15.; K = 10.;

BP = ContourPlot[∂y Detx[y] /. {y → x}, {γ1, 0, .2},
    {x, 0, 5}, Contours → {0}, ContourShading → False, ContourStyle → {Thick}];

XL = ContourPlot[γ1 == ((α + K β2) γ2 (α + δ1))/(β1 (Rin α γ2 + K δ2)), {γ1, 0, .2}, {x, 0, 5}, ContourStyle → {Dashed}];

NV = RegionPlot[Det0[x] > 0, {γ1, 0, .2}, {x, 0, 5}, PlotStyle → Gray];

Show[{BP, XL, NV}, FrameLabel → {"γ1 →", "x*→"}, ImageSize → Small]

Clear[α, β1, β2, γ1, γ2, δ1, δ2, Rin, K];
```




```
α = 1.; β1 = 1.95; β2 = 1.; γ1 = .1; γ2 = .1; δ1 = 1.; δ2 = 1.; Rin = 15.; K = 10.;

BP = ContourPlot[∂y Detx[y] /. {y → x}, {γ2, 0, .2},
    {x, 0, 5}, Contours → {0}, ContourShading → False, ContourStyle → {Thick}];

XL = ContourPlot[γ2 == (K β1 γ1 δ2)/(α^2 + K β2 δ1 + α (K β2 - Rin β1 γ1 + δ1)), {γ2, 0, .2}, {x, 0, 5}, ContourStyle → {Dashed}];

NV = RegionPlot[Det0[x] > 0, {γ2, 0, .2}, {x, 0, 5}, PlotStyle → Gray];

Show[{BP, XL, NV}, FrameLabel → {"γ2 →", "x*→"}, ImageSize → Small]

Clear[α, β1, β2, γ1, γ2, δ1, δ2, Rin, K];
```