Exercise 5:
The Lotka - Volterra cannibalism time budget model

Introduction

Resident dynamics:

$$\frac{d}{dt}R = r\,R\left(1-\frac{R}{K}\right) - \alpha\,R\sum_{j=1}^{k}\left(1-x_j\right)n_j$$

$$\frac{d}{dt}n_i = \epsilon\,\alpha\,R\,(1-x_i)\,n_i - \delta\,(1-x_i)\,n_i - (1-x_i)\,n_i\sum_{j=1}^{k}\beta\left(x_j\right)x_j\,n_j + \gamma\,\beta(x_i)\,x_i\,n_i\sum_{j=1}^{k}\left(1-x_j\right)n_j - \delta\,x_i\,n_i \qquad \text{for } i = 1, \ldots, k$$

Feedback environment:

$$\frac{d\boldsymbol{R}}{dt} = r\,\boldsymbol{R}\left(1-\frac{\boldsymbol{R}}{K}\right) - \alpha\,\boldsymbol{R}\cdot\sum_{j=1}^{k}\left(1-x_j\right)n_j$$

$$\frac{1}{\boldsymbol{n_i}}\frac{d\boldsymbol{n_i}}{dt} = \epsilon\,\alpha\,\boldsymbol{R}\,(1-\boldsymbol{x_i}) - \delta - (1-\boldsymbol{x_i})\sum_{j=1}^{k}\beta\left(x_j\right)x_j\,n_j + \gamma\,\beta(\boldsymbol{x_i})\,\boldsymbol{x_i}\sum_{j=1}^{k}\left(1-x_j\right)n_j$$

$$= \mathrm{A}(\boldsymbol{x_i}, \boldsymbol{E}) \qquad \text{for } i = 1, \ldots, k$$

Rewrite:

$$\frac{1}{\boldsymbol{n_i}}\frac{d\boldsymbol{n_i}}{dt} = \epsilon\,\alpha\,\boldsymbol{E_1}\,(1-\boldsymbol{x_i}) - \delta - (1-\boldsymbol{x_i})\,\boldsymbol{E_2} + \gamma\,\beta(\boldsymbol{x_i})\,\boldsymbol{x_i}\,\boldsymbol{E_3}$$

$$= \mathrm{A}(\boldsymbol{x_i}, \boldsymbol{E}) \qquad \text{for } i = 1, \ldots, k$$

$$\boldsymbol{E} = (\boldsymbol{E_1}, \boldsymbol{E_2}, \boldsymbol{E_3})$$

$$\frac{d\boldsymbol{E_1}}{dt} = r\,\boldsymbol{E_1}\left(1-\frac{\boldsymbol{E_1}}{K}\right) - \alpha\,\boldsymbol{E_1}\cdot\boldsymbol{E_3} \qquad \text{(resource density)}$$

$$\boldsymbol{E_2} = \sum_{j=1}^{k}\beta\left(x_j\right)x_j\,n_j \qquad \text{("cannibalism pressure")}$$

$$\boldsymbol{E_3} = \sum_{j=1}^{k}\left(1-x_j\right)n_j \qquad \text{("prey" density)}$$

Invader dynamics:

$$\frac{1}{m}\frac{dm}{dt} = \epsilon\,\alpha\,E_1\,(1-y) - \delta - (1-y)\,E_2 + \gamma\,\beta(y)\,y\,E_3$$

$$= \mathrm{A}(y, E)$$

Invasion fitness:

$$s_E(y) = \langle \mathrm{A}(y, E)\rangle$$

$$= \epsilon\,\alpha\,\langle E_1\rangle\,(1-y) - \delta\,\langle E_2\rangle + \gamma\,\beta(y)\,y\,\langle E_3\rangle$$

Effective dimension of E:

$R = E_1$ is assumed to be log – bounded

$$\Longrightarrow \boxed{0 = r\left(1 - \frac{\langle E_1\rangle}{K}\right) - \alpha\,\langle E_3\rangle} \Longleftrightarrow \langle E_1\rangle = K\left(1 - \frac{\alpha}{r}\,\langle E_3\rangle\right)$$

Hence, the effective dimension is 2, and

$$\boxed{s_E(y) = \epsilon\,\alpha\,K\,(1-y) - \delta\,\langle E_2\rangle + \left(\gamma\,\beta(y)\,y - \epsilon\,\alpha\,K\,\frac{\alpha}{r}\,(1-y)\right)\langle E_3\rangle}$$

and at most two resident types can coexist at a time;

Monomorphic resident population:

$$\boxed{\begin{array}{l}\langle E_2\rangle = \beta(x)\,x\,\langle n\rangle \\ \langle E_3\rangle = (1-x)\,n\end{array}} \Longrightarrow \boxed{\langle E_2\rangle = \frac{\beta(x)\,x}{1-x}\,\langle E_3\rangle}$$

$s_E(x) = 0$

$$\Longrightarrow\ 0 = \epsilon\,\alpha\,K\,(1-x) - \delta\,\frac{\beta(x)\,x}{1-x}\,\langle E_3\rangle + \left(\gamma\,\beta(x)\,x - \epsilon\,\alpha\,K\,\frac{\alpha}{r}\,(1-x)\right)\langle E_3\rangle$$

$$\Longrightarrow\ \langle E_3\rangle = \frac{K\,r\,(1-x)^2\,\alpha\,\epsilon}{r\,x^2\,\beta\,(\delta-(1-x)\,\gamma)+K\,(1-x)^2\,\alpha^2\,\epsilon}$$

$$\Longrightarrow\ \boxed{s_E^{\text{mono}}(y) = -\frac{K\,r\,(x-y)\,\alpha\,\beta\left(y\,\gamma+x\,(\gamma-2\,y\,\gamma)+x^2\,((-1+y)\,\gamma-\delta)\right)\epsilon}{r\,x^2\,\beta\,((-1+x)\,\gamma+\delta)+K\,(-1+x)^2\,\alpha^2\,\epsilon} = s_x\,(y)}$$

Dimorphic resident population:

$$0 = \epsilon\,\alpha\,K\,(1-x1) - \delta\,\langle E_2\rangle + \left(\gamma\,\beta\,(x1)\,x1 - \epsilon\,\alpha\,K\,\frac{\alpha}{r}\,(1-x1)\right)\langle E_3\rangle$$
$$0 = \epsilon\,\alpha\,K\,(1-x2) - \delta\,\langle E_2\rangle + \left(\gamma\,\beta\,(x2)\,x2 - \epsilon\,\alpha\,K\,\frac{\alpha}{r}\,(1-x2)\right)\langle E_3\rangle$$

$$\Longrightarrow\ \boxed{\begin{aligned}\langle E_2\rangle &= \frac{K\,r\,(x1+x2-x1\,x2)\,\alpha\,\beta\,\gamma\,\epsilon}{\delta\left(r\,(x1+x2)\,\beta\,\gamma+K\,\alpha^2\,\epsilon\right)}\\ \langle E_3\rangle &= \frac{K\,r\,\alpha\,\epsilon}{r\,(x1+x2)\,\beta\,\gamma+K\,\alpha^2\,\epsilon}\end{aligned}}$$

$$\Longrightarrow\ \boxed{s_E^{\text{di}}(y) = \frac{K\,r\,(x1-y)\,(x2-y)\,\alpha\,\beta\,\gamma\,\epsilon}{r\,(x1+x2)\,\beta\,\gamma+K\,\alpha^2\,\epsilon} = s_{x1,x2}(y)}$$

Questions

For the following default functions and parameter values :

$\alpha = 1;\ \ \gamma = 0.2;\ \ \delta = 0.1;\ \ \epsilon = 0.05;\ \ r = 1;\ \ K = 10;$

$\beta[x] = \beta 0 + \beta 1\,x^p$

with

$\beta 0 = 0.;\ \ \beta 1 = 1.5;\ \ p = 0.5;$

Produce:

- Pairwise invadability plot, and classify the singularities in terms of the "8 cases";
- Mutual invadability plot, and indicate the direction of evolution by calculating the sign of the selection gradients in the dimorphic population;
- Draw your conclusions with respect to the predicted evolutionary scenarios for different starting point for the monomorphic resident population.