Topology II Homework set 13
Spring 2015 Wed 29.4.2015

NB1: **The second course exam** will be on **Thursday, May 7th, at 13:00–15:00** in one of the lecture halls in Exactum: check the entrance to the Lars Ahlfors Auditorium (A111) for the exact location. The exam covers chapters 9–13, 15–19, and 20.1–20.3. of the Topologia II textbook, excluding all the supplementary material marked with a star, "*". More details will be given on the course web page roughly one week before the exam.

NB2: The last tutorials will be held on **Wednesday, April 29th, at 12–14 in the lecture hall DK118**. The last lecture will be held on Tue 28th, as usual, and it will include a review of the course material.

NB3: Exercise 6* is a voluntary bonus exercise.

Exercise 1

Consider some given topological spaces $X_j \neq \emptyset$, $j \in J$. Show that, if $X := \prod_{j\in J} X_j$ is compact, then every X_j, $j \in J$, has to be compact.

Exercise 2

Suppose $X \neq \emptyset$ is a compact Hausdorff space and $f : X \to X$ is continuous. Show that there is a *closed nonempty* $A \subset X$ for which $fA = A$.

(*Hint:* Consider the iterated images $fX, ffX, fffX, \ldots$)

Exercise 3

Suppose X is a normal connected space with more than one point, $\#X \geq 2$. By applying Urysohn's lemma show that then necessarily $\operatorname{card} X \geq \operatorname{card} \mathbb{R}$.

Remark: There are countable connected Hausdorff spaces which by the above result cannot be normal.

Exercise 4

Consider the open bounded interval $Y :=]a, b[$, $-\infty < a < b < \infty$, endowed with the ordinary topology. Suppose X is a metrizable space, A is a closed subset of X and $f : A \to Y$ is continuous. Show that f has a continuous extension $g : X \to Y$.

Remark: The result shows that open intervals are *absolute retracts*. (Chapter 20 in the textbook has for more discussion about the topic.)

(*Hint:* The Tietze extension theorem yields an extension $g_1 : X \to [a, b]$. Find an extension with the right properties by defining $g(x) = g_1(x)h(x)$, $x \in X$, where the function h is obtained from a suitable application of the Urysohn's lemma.)

(Continues. . .)

Exercise 5

Let $I := [0,1]$ denote the closed unit interval endowed with the ordinary topology. Define $X \subset I^I$ as the collection of increasing functions: that is, X contains those functions $f : I \to I$ for which $f(s) \le f(t)$ whenever $s \le t$.

Let I^I have the product topology and X the inherited relative topology.

(a) Show that X is compact.

(b) Let $A := \{f_a \,|\, a \in I\}$ where f_a denotes the "step function at a" for a point $a \in I$. Explicitly, f_a is defined by

$$f_a(t) := \begin{cases} 0, & \text{for } t < a\,, \\ \frac{1}{2}, & \text{for } t = a\,, \\ 1, & \text{for } t > a\,. \end{cases}$$

Conclude that $A \subset X$. Show that A is *not* separable.

(c) Is X metrizable? (*Hint:* Theorem 12.21.)

Exercise 6* (bonus exercise)

Consider a set X with two topologies $\mathcal{T}$ and $\mathcal{T}_1 \subset \mathcal{T}$. Show that if $(X, \mathcal{T})$ is compact and $(X, \mathcal{T}_1)$ is Hausdorff, then necessarily $\mathcal{T}_1 = \mathcal{T}$.

Conclude that compact Hausdorff topologies are *minimal Hausdorff topologies*: any strictly coarser topology can no longer be Hausdorff.

(*Hint:* Apply Theorem 15.18 to the map $\mathrm{id} : (X, \mathcal{T}) \to (X, \mathcal{T}_1)$.)