Topology II Homework set 9

Spring 2015 Tue 24.3.2015

NB: The results for the first midterm exam are now available on the department's results webpage (link available on the course homepage). If you wish, you can check the grading of your answers from the lecturer.

Exercise 1

Prove Theorem 10.14 (Weierstrass M-test):

Suppose D is a set, E is a Banach space, and $u_n : D \to E$, $n \in \mathbb{N}$, are functions. Assume that there are $M_n \geq 0$ are such that $\sum_n M_n < \infty$ and $\|u_n(x)\| \leq M_n$ for all $x \in D$, $n \in \mathbb{N}$. Prove that then the series $\sum_n u_n$ converges uniformly on D.

(*Hint:* Denote $R_n := \sum_{j=n+1}^{\infty} M_j$ and $s_n(x) := \sum_{j=1}^{n} u_j(x)$. Show first that $\|s_{n+p}(x) - s_n(x)\| \leq R_n$, for all $x \in D$ and $p, n \in \mathbb{N}$, and then use completeness.)

Exercise 2

A subset A of a topological space X is called *nowhere dense* (suom. *harva*) if $\operatorname{int} \overline{A} = \emptyset$. A subset is called *meager* (suom. *laiha*) or *of first category* if it can be written as a countable union of nowhere dense sets.

Prove the following statements:

(a) A countable union of meager sets is meager.
(b) If X is a complete metric space and $A \subset X$ is meager, then $\operatorname{int} A = \emptyset$.
(c) Give an example of a meager set $A \subset \mathbb{R}$ which is *not* nowhere dense.

Exercise 3

Consider a set $X \neq \emptyset$ and a metric space (Y, d). Denote the collection of all bounded functions $X \to Y$ by "$\operatorname{raj}(X, Y)$". As proven in "Topologia I, Lause I.2.14" a function $f : X \to Y$ belongs to $\operatorname{raj}(X, Y)$ if and only if there are $y \in Y$ and $r > 0$ such that $d(f(x), y) < r$ for all $x \in X$.

(a) Show that the equation

$$e(f, g) := \sup_{x \in X} d(f(x), g(x)),$$

defines a metric on $\operatorname{raj}(X, Y)$. (This is called the *sup-metric* on $\operatorname{raj}(X, Y)$.)

(b) Prove that, if Y is complete, then $\operatorname{raj}(X, Y)$ is complete in the sup-metric e.

(Continues. . .)

Exercise 4

Consider a *smooth function* $f : \mathbb{R} \to \mathbb{R}$ (it has derivatives $f^{(n)}(x)$ at all orders $n \in \mathbb{N}$ and at every point $x \in \mathbb{R}$). Assume that to every point $x \in \mathbb{R}$ we can find some order $n(x) \in \mathbb{N}$ such that the corresponding derivative vanishes at x, that is, such that $f^{(n(x))}(x) = 0$. Prove that then any interval $[a, b]$, $a < b$, contains an open interval on which f is a polynomial function.

(*Hint:* Use Baire's theorem and the sets $A_k := \{x \in \mathbb{R} \,|\, f^{(k)}(x) = 0\}$, $k \in \mathbb{N}$. In fact, f is a polynomial function everywhere, but the proof of this property is more involved: see Exercise 10:15 in the textbook if you are interested in the details.)

Exercise 5

As usual, let $\bar{B}^n := \{x \in \mathbb{R}^n \,|\, |x| \leq 1\}$ and $S^{n-1} := \{x \in \mathbb{R}^n \,|\, |x| = 1\}$ denote the closed unit ball and the unit sphere in $\mathbb{R}^n$. Show that $\bar{B}^n/S^{n-1} \approx S^n$ by constructing a continuous function $f : \bar{B}^n \to S^n$ for which $\bar{B}^n/S^{n-1}$ is the partition corresponding to R_f in the canonical decomposition of f.

(*Hint:* The proof for the case $n = 1$ is essentially the same as the proof of $[0, 1]/\{0, 1\} \approx S^1$ given in Example 9.12.1 of the textbook and in the lectures.)