Topology II Homework set 8
Spring 2015 Tue 17.3.2015

Change of schedule: The **lecture on Tuesday, March 17th** will start two hours earlier than usually: it will be held at **14–16** in the usual place, lecture hall C124.

Exercise 1

Let I denote the closed interval $[0,1]$. Show that $\mathbb{R}/I \approx \mathbb{R}$.

(*Reminder:* If X is a topological space and $A \subset X$ is nonempty, the notation "X/A" denotes the quotient space which is obtained from the partition consisting of A and the singlets $\{x\}$, $x \notin A$. Thus X/A can be thought of as the space obtained by *collapsing* A *into a point*.)

Exercise 2

(a) Suppose X is a Hausdorff space. Show that $\{x\}$ is closed for every $x \in X$.

(b) Consider a Hausdorff space X and $A \subset X$. Show that the relative topology inherited by A from X is also Hausdorff.

(c) Consider some topological space X and $A \subset X$. Show that if $\{A\}$ is closed in the quotient space X/A, then A is closed in X.

(d) Consider $A :=]0,1[\subset \mathbb{R}$. Show that the quotient space $\mathbb{R}/A$ is *not* Hausdorff and conclude from this that $\mathbb{R}/A \not\approx \mathbb{R}$.

Exercise 3

The *cone* formed from a topological space X is the quotient space $c(X) := (X \times I)/(X \times \{1\})$ where $I := [0,1]$ is endowed with the ordinary topology. Show that $c(S^{n-1}) \approx \bar{B}^n$.

(*Hint:* Find a suitable map whose canonical decomposition allows concluding the result from Theorem 9.10; it is probably a good idea to begin with the case $n = 1$. Note also that $S^{n-1} \times I$ is easily seen to be homeomorphic to a compact subset of $\mathbb{R}^{n+1}$ by relying on Theorem 7.14.)

(Continues. . .)

Exercise 4

Let R be an equivalence relation on the space X, and S an equivalence relation on the space Y. Suppose $f : X \to Y$ is a function such that

$$xRx' \quad \Rightarrow \quad f(x)Sf(x'),$$

for all $x, x' \in X$.

(a) Show that there is a unique function $\hat{f} : X/R \to Y/S$, for which $\hat{f} \circ p_R = p_S \circ f$.
(b) Prove that $\hat{f}$ is continuous if f is continuous.

Exercise 5

Prove that the *projective plane* I^2/R defined in Example 9.12.6 is homeomorphic to the *projective space* P^2 defined in the lectures and in Example 9.5.

Hints:

(a) Explicitly, the projective plane is defined by using the following equivalence relation R on the unit square $I^2 := [0,1]^2$: we set

$$(x,y)R(x,y), \quad (0,y)R(1,1-y), \quad \text{and} \quad (x,0)R(1-x,1), \quad \text{for all } x,y \in [0,1],$$

and also set symmetrically $(1,1-y)R(0,y)$ and $(1-x,1)R(x,0)$. The corresponding partition consists of the singlets $\{(x,y)\}$, with $0 < x,y < 1$, and the doublets $\{(0,y),(1,1-y)\}$, with $0 \le y \le 1$, and $\{(x,0),(1-x,1)\}$, with $0 < x < 1$. The projective space is $P^2 := S^2/R'$ where $\mathbf{x}R'\mathbf{x}$ and $\mathbf{x}R'(-\mathbf{x})$ for all $\mathbf{x} \in S^2$.

(b) As the first step, prove that $I^2 \approx \bar{B}^2$. (The following gives an outline for a proof: Move the center of the square to the origin and then consider the map $\rho : \mathbf{x} \mapsto \mathbf{x}/|\mathbf{x}|$ on the boundary of the square. Show that $\Phi(\mathbf{x}) := |\mathbf{x}|\rho^{-1}(\mathbf{x}/|\mathbf{x}|)$ defines a homeomorphism from $\bar{B}^2$ to the square. (*Hint:* What do you know about continuous bijections with a compact domain?))

(c) Find a function $f : \bar{B}^2 \to S^2$ which has $f(x,y) = (x,y,0)$, whenever $(x,y) \in S^1$, and which is an embedding of $\bar{B}^2$ into the upper half plane $S^2_+ := \{\mathbf{x} \in S^2 \mid \mathbf{x}_3 \ge 0\}$.

(d) Use the previous exercise.