Topology II Homework set 7
Spring 2015 Tue 10.3.2015

Reminder: The first course exam is on Thursday, March 5th, at 13:00–15:00.

NB: There will be no lectures nor tutorials on the exam week. However, solving these exercises could be quite useful for preparing to the exam.

Exercise 1

Let X be a nonempty product space $\prod_{j\in J} X_j$ with an indexing set $J \neq \emptyset$. For some $a \in X$ and $k \in J$, consider the corresponding "a-sheet over X_k" which is defined as

$$A := \{x \in X \mid x_j = a_j \text{ if } j \in J, j \neq k\} .$$

Show that $A \approx X_k$.

Exercise 2

Let $g, h : \mathbb{R} \to \mathbb{R}$ be *arbitrary* functions. Consider the product space $X := \mathbb{R}^{\mathbb{R}}$ and the function $F : X \to X$ defined by the formula $F(x)_t := g(t)x_{h(t)}$, $t \in \mathbb{R}$. Prove that F is continuous.

(*Hint:* Theorem 7.10.)

Exercise 3

Consider topological spaces X_1, X_2 and their subsets $A_1 \subset X_1$, $A_2 \subset X_2$. Show that the following equalities hold for $A_1 \times A_2 \subset X_1 \times X_2$:

$$\partial(A_1 \times A_2) = (\partial A_1 \times \overline{A_2}) \cup (\overline{A_1} \times \partial A_2) ,$$
$$\operatorname{int}(A_1 \times A_2) = (\operatorname{int} A_1) \times (\operatorname{int} A_2) .$$

Exercise 4

Suppose X is a topological space and $A \subset X$. As in Exercise 6.5, denote the characteristic function of A by χ_A. Show that

$$\partial A = \{x \in X \mid \chi_A \text{ is discontinuous at } x\} .$$

(Continues. . .)

Exercise 5

Consider a topological space $(X, \mathcal{T})$ and a nonempty collection $\mathcal{A}$ of subsets of X. Let each $A \in \mathcal{A}$ have the relative topology $\mathcal{T}|_A$ inherited from X. Then the collection of inclusion maps $j_A : A \hookrightarrow X$, $A \in \mathcal{A}$, coinduces a topology $\mathcal{T}'$ on X.

(a) Show that $\mathcal{T} \subset \mathcal{T}'$.

(b) Prove that $\mathcal{T}'|_A = \mathcal{T}|_A$ for every $A \in \mathcal{A}$.

(c) Construct an example in which $\mathcal{T}' \neq \mathcal{T}$.

(d) Show that $\mathcal{T}' = \mathcal{T}$, if $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$, and $\mathcal{A}$ is the collection of all compact subsets of X.