Topology II Homework set 5
Spring 2015 Tue 17.2.2015

NB: The first course exam is planned for **Thursday, March 5th, at 13:00–15:00**. *Please send an e-mail to the lecturer as soon as possible if you wish to take the exam but cannot come at the above time.* (Let me then also know about which times on the same week, 2.–6.3., would be possible for you.)

NB: Exercise 6* is again a voluntary bonus exercise.

Exercise 1

A family of functions $f_j : X \to Y_j$, $j \in J$, *separates points in* X if every $x, y \in X$, $x \neq y$, has some $j \in J$ for which $f_j(x) \neq f_j(y)$.

Suppose that X is a set and $J \neq \emptyset$ is an indexing set for *Hausdorff* spaces Y_j and functions $f_j : X \to Y_j$, $j \in J$. Show that the topology induced by the family $(f_j)_{j \in J}$ on X is Hausdorff if and only the family separates points on X

Exercise 2

Let $\mathcal{T}_{f,g}$ denote the topology induced on $\mathbb{R}$ by two given functions $f, g : \mathbb{R} \to \mathbb{R}$ where the target space has the ordinary topology. Is $\mathcal{T}_{f,g}$ Hausdorff when $f(x) := x^2$, $x \in \mathbb{R}$, and

(a) $g(x) := -x^2$, $x \in \mathbb{R}$?
(b) $g(x) := (x-1)^2$, $x \in \mathbb{R}$?

(Remember to prove your answer.)

Exercise 3

The *graph* of a function $f : X \to Y$ is defined as the following subset of $X \times Y$

$$\mathcal{G}(f) := \{(x, f(x)) \,|\, x \in X\} \ .$$

Suppose that $f : X \to Y$ is continuous and let $\mathcal{G}(f)$ have the relative topology inherited from the product space $X \times Y$. Show that $\mathcal{G}(f)$ is homeomorphic to X.

Exercise 4

Let X, Y and Z be topological spaces. Show that $X \times Y \times Z \approx (X \times Y) \times Z$.

(*Alternative, more general, version of the exercise:* An analogous result holds for general product spaces $X = \prod_{j \in J} X_j$ and partitions K_α, $\alpha \in A$, of J into disjoint nonempty subsets such that $\cup_{\alpha \in A} K_\alpha = J$. Formulate and prove this result.)

(Continues...)

Exercise 5

Let X be a topological space. Show that the collection of all continuous functions $f : X \to X$ induces the original topology on X.

Exercise 6* (bonus exercise)

Weak-* topology

Let E be a real normed space and denote its dual space by E^*. (E^* is defined as the collection of all continuous linear maps $E \to \mathbb{R}$.) For any $x \in E$, the formula $f_x(\varphi) := \varphi(x)$, $\varphi \in E^*$, defines a function $f_x : E^* \to \mathbb{R}$. Therefore, the family of functions $(f_x)_{x\in E}$ induces a topology on E^* from the ordinary topology on $\mathbb{R}$. This induced topology is called the *weak-* topology* (suom. *heikko tähtitopologia*).

(a) Show that the weak-* topology is always Hausdorff.
(*Hint:* Exercise 1.)

(b) Consider some topological space X and a map $g : X \to E^*$. Show that g is weak-* continuous if and only if the functions $t \mapsto (g(t))(x)$ from X to $\mathbb{R}$ are continuous in the ordinary topology for all $x \in E$.
(*Hint:* Universality.)

(c) If $r > 0$, $F \subset E$, and $h : F \to \mathbb{R}$, define

$$B_F(h,r) := \{\varphi \in E^* \,|\, |\varphi(x) - h(x)| < r \text{ for all } x \in F\} \ ,$$

and $\mathcal{B} := \{B_F(h,r) \,|\, r > 0, F \subset E,\ F \text{ finite},\ h : F \to \mathbb{R}\}$. Show that the collection $\mathcal{B}$ is a base for the weak-* topology on E^*.
(*Hint:* Theorems 2.4 and 6.8.)

Remark 1: Despite the similarities in (c) to the definition of metric topologies, the weak-* topology is typically *not* metrizable.

Remark 2: It has been proven in the Topology I textbook (Lause 15.21) that

$$\|\varphi\|_{E^*} := \sup_{x\in E, \|x\|\le 1} |\varphi(x)|$$

defines a norm on the vector space E^*. The metric topology determined by this norm is called the *norm topology* on E^*. As explained in the course textbook at 6.4.3, each of the functions f_x, $x \in E$, is actually linear and bounded in the above norm. Thus every f_x is continuous when E^* is endowed with the norm topology. This implies that the weak-* topology is always coarser than the norm topology on E^*.