Topology II Homework set 3

Spring 2015 Tue 3.2.2015

NB: Exercise 6* is again a voluntary bonus exercise.

Exercise 1

Consider a function $f : X \to Y$ and two families of subsets $A_j \subset X$ and $B_j \subset Y$, $j \in J$, indexed by a set J. Prove the following relations for preimages and images involving unions and intersections:

(a) $f^{\leftarrow}\big[\bigcup_{j\in J} B_j\big] = \bigcup_{j\in J} f^{\leftarrow}[B_j]$.
(b) $f^{\leftarrow}\big[\bigcap_{j\in J} B_j\big] = \bigcap_{j\in J} f^{\leftarrow}[B_j]$.
(c) $f\big[\bigcup_{j\in J} A_j\big] = \bigcup_{j\in J} f[A_j]$.
(d) $f\big[\bigcap_{j\in J} A_j\big] \subset \bigcap_{j\in J} f[A_j]$.
(e) Show that, if f is injective and $J \neq \emptyset$, then $f\big[\bigcap_{j\in J} A_j\big] = \bigcap_{j\in J} f[A_j]$.
(f) Suppose f is *not* injective. Find some $A, A' \subset X$ for which $f[A \cap A'] \neq f[A] \cap f[A']$.

Exercise 2

Suppose $f : X \to Y$ is a homeomorphism between topological spaces X and Y. Prove that then every subset $A \subset X$ satisfies:

(a) $\overline{f[A]} = f\big[\,\overline{A}\,\big]$.
(b) $\operatorname{int} f[A] = f[\operatorname{int} A]$.
(c) $\operatorname{ext} f[A] = f[\operatorname{ext} A]$.
(d) $\partial f[A] = f[\partial A]$.

Remark: Therefore, all "topological properties" are preserved in a homeomorphism.
Hint: This is not a lengthy exercise *if* you use the Theorems of Sections 1–3 of the textbook.

Exercise 3

Define a function $f : \mathbb{R} \to \mathbb{R}$ using the formula $f(x) := -x$. By Section 3.6 of the textbook, f is continuous in the ordinary topology. (If you are unsure about the reasoning used in Section 3.6, recall more details from the Topology I textbook.)

Is f continuous if both the domain and the target set are endowed with the topology $\mathcal{T}_{\text{pa}}$ defined in Exercise 2.2? (Remember to prove your answer.)

Exercise 4

Consider some $n, p \in \mathbb{N}$ and a function $f : \mathbb{R}^n \to \mathbb{R}^p$ which is continuous using the ordinary topologies. Prove that, if $|f(x_n)| \to \infty$ whenever (x_n) is a sequence such that $|x_n| \to \infty$, then f is a closed map. Conclude that every polynomial $\mathbb{R} \to \mathbb{R}$ is a closed map.

(*Hint:* The solution requires basic results about compact sets and continuity in the normed space $\mathbb{R}^n$, as (hopefully) familiar from Topology I.)

(Continues. . .)

Exercise 5

Define $\mathcal{T} := \{\emptyset\} \cup \{U \subset \mathbb{R} \,|\, \mathbb{R} \setminus U \text{ is countable}\}$. Show that $\mathcal{T}$ is a topology on $\mathbb{R}$ and denote the resulting topological space by X. Prove the following statements:

(a) A sequence (x_n) in X converges to a point a if and only if there is some $n_0 \in \mathbb{N}$ such that $x_n = a$ for all $n \geq n_0$.
(b) The point 0 belongs to the closure of the interval $A := [1, 2]$ in X but no sequence in A converges to 0.
(c) Consider an arbitrary function $f : X \to Y$ from X to a topological space Y. Show that, if a sequence $x_n \to a$, then $f(x_n) \to f(a)$.
(d) Find a discontinuous function $f : X \to Y$ to some topological space Y.

Remark: This shows that, unlike in metric spaces (see Topology I.11.8), the convergence condition about sequences given in item (c) does not guarantee that the function is continuous in the topology $\mathcal{T}$.

Exercise 6* (bonus exercise)

Filter bases and continuity of functions

Consider a topological space X. A collection $\mathcal{F} \subset \mathcal{P}(X)$ is called a *filter base* if it satisfies the following conditions:

(1) $\mathcal{F} \neq \emptyset$.
(2) $\emptyset \notin \mathcal{F}$.
(3) If $A, B \in \mathcal{F}$, then $C \subset A \cap B$ for some $C \in \mathcal{F}$.

We say that a filter base $\mathcal{F}$ *converges to a point* $a \in X$ if every neighborhood U of a has some $A \in \mathcal{F}$ for which $A \subset U$.

Consider then a function $f : X \to Y$ between the topological spaces X and Y. Prove the following:

(a) If $a \in X$, then the neighborhoods of a form a filter base which converges to a.
(b) If $\mathcal{F}$ is a filter base in X, then the collection $\{fA \,|\, A \in \mathcal{F}\}$ is a filter base in Y.
(c) If f is continuous at $a \in X$ and $\mathcal{F}$ is a filter base which converges to a, then $\{fA \,|\, A \in \mathcal{F}\}$ converges to $f(a)$.
(d) Also the converse of (c) holds: Suppose $a \in X$ is such that for every filter base $\mathcal{F}$ converging to a also the filter base $\{fA \,|\, A \in \mathcal{F}\}$ converges to $f(a)$. Show that f is then continuous at a.