Topology II Homework set 2

Spring 2015 Tue 27.1.2015

NB: Exercise 6* is again a voluntary bonus exercise.

Exercise 1

Let d and e be metrics on the set X and denote the topologies they generate on X by $\mathcal{T}_d$ and $\mathcal{T}_e$.

(a) Show that, if there is a constant $c > 0$ such that $d(x, y) \leq ce(x, y)$ for all $x, y \in X$, then $\mathcal{T}_d \subset \mathcal{T}_e$.

(b) Two metrics d and e on X are called *strongly equivalent* if there are $a, b > 0$ such that

$$ad(x, y) \leq e(x, y) \leq bd(x, y)\,, \quad \text{for all } x, y \in X\,.$$

Show that strongly equivalent metrics are also topologically equivalent: then $\mathcal{T}_d = \mathcal{T}_e$.

Exercise 2

Consider the collection $\mathcal{B} := \{\,[a, b[\;|\, a, b \in \mathbb{R}, a < b\}$ of intervals of $\mathbb{R}$.

(a) Prove that $\mathcal{B}$ is a base for a topology $\mathcal{T}_{\text{pa}}$ on $\mathbb{R}$. Show that it is strictly finer than the ordinary topology on $\mathbb{R}$.
(*Hint:* See Example 2.11. in the textbook.)

(b) Prove that any interval $B \in \mathcal{B}$ is both open and closed in $(\mathbb{R}, \mathcal{T}_{\text{pa}})$.

(c) Determine the closure and boundary of the interval $]0, 1[$ in $(\mathbb{R}, \mathcal{T}_{\text{pa}})$.

Exercise 3

Consider the following three collections $\mathcal{B}_i \subset \mathcal{P}(\mathbb{R})$, $i = 1, 2, 3$:

(1) $\mathcal{B}_1 := \{\,]x - 1, x + 1[\;|\, x \in \mathbb{R}\}$,

(2) $\mathcal{B}_2 := \{\,[-\frac{1}{n}, \frac{1}{n}]\,|\, n \in \mathbb{N}\}$,

(3) $\mathcal{B}_3 := \{\,]a, b[\,\cup\,]c, \infty[\,|\, a < b < c\}$.

Which of these collections yield a base for some topology on $\mathbb{R}$? For each i for which $\mathcal{B}_i$ is a base, let $\mathcal{T}$ denote the corresponding topology generated by $\mathcal{B}_i$.

(a) Is there is any relation between $\mathcal{T}$ and the ordinary topology on $\mathbb{R}$? (coarser, finer, equal?)

(b) Determine $\text{cl}_{\mathcal{T}}\,]0, 1[$.

Exercise 4

What is the topology on $\mathbb{R}^3$ generated by the collection of all planes in $\mathbb{R}^3$?

(Continues. . .)

Exercise 5

Suppose X is topological space and $A \subset X$. Prove the following statements:

(a) $\partial\partial A \subset \partial A$.
(b) If A is closed, then $\partial\partial A = \partial A$.
(c) Always $\partial\partial\partial A = \partial\partial A$.

By considering the case $X := \mathbb{R}$, $A := \mathbb{Q}$, show that in general the inclusion "$\subset$" cannot be replaced by an equality "$=$" in item (a).

Exercise 6* (bonus exercise)

Order topology

A relation "$\le$" on a set H is called an *order*, if it has the following properties for every $a, b, c \in H$:

(1) $a \le a$.
(2) If $a \le b$ and $b \le a$, then $a = b$. (Often shortened to "$a \le b \le a \Rightarrow a = b$.")
(3) If $a \le b$ and $b \le c$, then $a \le c$. (Often shortened to "$a \le b \le c \Rightarrow a \le c$.")

An order is *total*, if additionally every $a, b \in H$ has $a \le b$ or $b \le a$. If $a \le b$ and $a \ne b$, we denote $a < b$. We can then also define "intervals on H" analogously to those in $\mathbb{R}$: we define $[a, b] := \{x \in H \,|\, a \le x \le b\}$, whenever $a \le b$, and using this notation we define for all $a < b$,

$$]a, b[:= [a, b] \setminus \{a, b\}\,, \quad]a, b] := [a, b] \setminus \{a\}\,, \quad [a, b[:= [a, b] \setminus \{b\}\,.$$

(Note that it is possible that $]a, b[= \emptyset$, even though $a < b$.)

Assume that $(H,\le)$ is a totally ordered set with at least two elements. Define a collection of its subsets as $\mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$ where

(1) $\mathcal{B}_0 := \{\,]a, b[\;|\; a < b\}$.
(2) $\mathcal{B}_1 := \{\,]a, M] \;|\; a \in H, a \ne M\}$ if H contains a *maximum* M (this is an element such that $a \le M$ for all $a \in H$). Otherwise, define $\mathcal{B}_1 := \emptyset$.
(3) $\mathcal{B}_2 := \{\, [m, b[\;|\; b \in H, b \ne m\}$ if H contains a *minimum* m (this is an element such that $m \le a$ for all $a \in H$). Otherwise, define $\mathcal{B}_2 := \emptyset$.

Prove that then the following statements hold:

(a) Suppose that $a_1 \le b_1$ and $a_2 \le b_2$. Set $a := a_1$, if $a_1 \ge a_2$, and else set $a := a_2$. (Then $a = \max(a_1, a_2)$). Similarly, set $b := b_1$, if $b_1 \le b_2$, and else $b := b_2$. (Then $b = \min(b_1, b_2)$). Show that

$$[a_1, b_1] \cap [a_2, b_2] = \begin{cases} [a, b]\,, & \text{if } a \le b\,, \\ \emptyset\,, & \text{otherwise}\,. \end{cases}$$

(b) Prove that $\mathcal{B}$ is a base for a topology on H. This topology is called the *order topology* generated by the order $\le$ on H and we denote it by $\mathcal{T}_{\le}$.
(*Hint:* Recall item (a) and that $A \setminus B = A \cap \complement B$ for $A, B \subset H$.)
(c) Show that, if $a \in H$, then the sets $\{x \in H \,|\, x < a\}$ and $\{x \in H \,|\, x > a\}$ are open in the order topology.
(d) Prove that $(H, \mathcal{T}_{\le})$ is a Hausdorff space.