Topology II Homework set 1

Spring 2015 Tue 20.1.2015

NB: Exercise 6* is a voluntary bonus exercise. In particular, this means that it is possible to get in total 10 points from Homework sets 0 and 1, although the "maximum" points are 5.

Exercise 1

Prove the following statements, assuming only that X is a topological space.

(a) If $A, B \subset X$, then $\overline{A \cup B} = \overline{A} \cup \overline{B}$. (Item (8) in "Lause 1.12." in the textbook.)

(b) For any $A \subset X$, it holds that $\partial A = \overline{A} \cap \overline{\complement A} = \overline{A} \setminus \operatorname{int} A$. Conclude that ∂A is always a closed set. (Item (5) in "Lause 1.14." in the textbook.)

(c) If X is metrizable, it is a Hausdorff space.

(*Hint:* You can find all proofs in the textbook of the "Topologia I" course. The purpose of the exercise is to find and read the proofs from the book, and to check that they are valid *also in the present case*, in which the topology of X is not necessarily given by a metric.)

Exercise 2

Let $\mathcal{T} \subset \mathcal{P}(\mathbb{R})$ to contain $\emptyset$, $\mathbb{R}$, and $]a, \infty[$ for $a \in \mathbb{R}$. (Here, $]a, \infty[:= \{x \in \mathbb{R} \mid x > a\}$.)

(a) Show that $\mathcal{T}$ is a topology on $\mathbb{R}$.

(b) Is $\mathcal{T}$ coarser, finer, or equal to the ordinary topology on $\mathbb{R}$?

(c) Compute $\overline{[0, 1]}$ in the topological space $(\mathbb{R}, \mathcal{T})$.
(*Hint:* What are the closed subsets in this topology?)

Exercise 3

Discrete topology

Suppose $X \neq \emptyset$ and define $\mathcal{T}_{\text{dis}} := \mathcal{P}(X) = \{A \mid A \subset X\}$.

(a) Show that $\mathcal{T}_{\text{dis}}$ is a topology on X.

(b) Define a function $d : X \times X \to \mathbb{R}_+$ by the following rules: $d(x, y) = 0$, if $x = y$, and $d(x, y) = 1$, if $x \neq y$. Show that d is a metric and that $\mathcal{T}_d = \mathcal{T}_{\text{dis}}$, i.e., that the metric topology generated by d is equal to $\mathcal{T}_{\text{dis}}$.

(c) In "Topologia I", a *metric* space was called "compact" if every sequence in it has a convergent subsequence. Show that now $A \subset X$ is compact if and only if $\#A < \infty$.

Remark: If $X = \emptyset$, we have $\mathcal{T}_{\text{dis}} := \mathcal{P}(X) = \{\emptyset\}$. Hence, then $\mathcal{T}_{\text{dis}} = \mathcal{T}_{\text{mini}}$, and by Homework 0.3 also this special case results in a topology on X. This topology is metrizable since the empty map $d : \emptyset \to \mathbb{R}_+$ is a metric on $\emptyset$. (Note that $\emptyset \times \emptyset = \emptyset$.)

(Continues on the other side. . .)

Exercise 4

Consider a function $f : X \to Y$ between sets X, Y.

(a) Show that $f^{\leftarrow}[f[A]] \supset A$ for all $A \subset X$. Prove that $f^{\leftarrow}[f[A]] = A$ if f is injective. ("Injective" means one-to-one.)

(b) Show that $f[f^{\leftarrow}[B]] = B \cap f[X] \subset B$ for all $B \subset Y$. Therefore, $f[f^{\leftarrow}[B]] = B$ if f is surjective. ("Surjective" means onto.)

Remark: The notations have been explained at Homework 0.2.

Exercise 5

Let X be a topological space and $A \subset X$. Prove that the following statements are *equivalent*:

(a) A is dense in X.

(b) A meets every open nonempty subset of X.

(c) $\operatorname{int} \complement A = \emptyset$.

Exercise 6*

(voluntary bonus exercise)

Two alternative ways to define a topological space are explained in the textbook, at subsection 1.16. The point of this exercise is to prove that the second of these, using the axioms of K. Kuratowski, is equivalent to the now standard definition used in the textbook.

If $\mathcal{T}$ is a topology on a set X, Theorem 1.12. in the textbook implies that the mapping $A \mapsto \overline{A}$ has the following properties:

(K1) $A \subset \overline{A}$.

(K2) $\overline{\overline{A}} = \overline{A}$.

(K3) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(K4) $\overline{\emptyset} = \emptyset$.

(By Theorem 1.8., $\emptyset$ is closed, and thus (K4) follows by applying item (6) in Theorem 1.12.)

Prove that the converse result is also true: if X is a set and $A \mapsto \overline{A}$ is a function $\mathcal{P}(X) \to \mathcal{P}(X)$ which satisfies all of the axioms (K1)–(K4), then the collection

$$\mathcal{T} := \left\{ V \subset X \,\middle|\, \overline{\complement V} = \complement V \right\}$$

is a topology on X and the closure of every $A \subset X$ in the topology $\mathcal{T}$ is equal to $\overline{A}$.