Department of Mathematics and Statistics
Stochastic processes on domains
Suggestions to excercise problem sheet 6

Note. In the Problems 1-12 the j,k and n are always integers.
Note. The Problems 1-4 provide a proof of the Lévy's Characterisation Theorem.

1. Suppose M is a 1-dimensional continuous local $(\mathscr{F}_t)$-martingale. Show that for every $\lambda \in \mathbb{C}$, the process

$$Z_t^\lambda := \exp\Big(\lambda M_t - \tfrac{1}{2}\lambda^2 \langle\, M, M\,\rangle_t\Big)$$

is a local martingale. (Hint. Itō with f such that $f(M, \langle\, M, M\,\rangle) = Z^\lambda$).
Suggestion. Let's denote $A_t = \langle\, M, M\,\rangle_t$. This is a continuous increasing, $(\mathscr{F}_t)$-adapted process.

Following the hint, let's take $f(x, y) = e^{\lambda x - \frac{1}{2}\lambda^2 y}$. Then $\partial_x f = \lambda f$, $\partial_x^2 f = \lambda^2 f$ and $\partial_y f = -\frac{1}{2}\lambda^2 f$. Therefore,

$$\begin{aligned} f(M_t, A_t) &= f(M_0, A_0) + \int_0^t \lambda f(M_s, A_s)\,\mathrm{d}M_s - \int_0^t \tfrac{1}{2}\lambda^2 f(M_s, A_s)\,\mathrm{d}A_s \\ &\quad + \int_0^t \tfrac{1}{2}\lambda^2 f(M_s, A_s)\,\mathrm{d}\langle\, M, M\,\rangle_s \end{aligned}$$

Since $A = \langle\, M, M\,\rangle$ the two terms on the right cancel each other and we are left with a local martingale term $\lambda f(M_t, A_t)\,\mathrm{d}M_t$. This gives the claim.

2. Suppose $X = (X^1, \dots, X^d)$ is a d-dimensional continuous local $(\mathscr{F}_t)$-martingale, suppose $X_0 = 0$ and suppose $\langle\, X^j, X^k\,\rangle_t = [\,j = k\,]t$. Show that for every $f = (f_1, \dots, f_d)$ with $f_j \in L^2(\mathbb{R}^+)$ the process

$$Y_t^f := \exp\Big(i \sum_{k=1}^d \int_0^t f_k(X_s)\,\mathrm{d}X_s^k + \tfrac{1}{2}\sum_{k=1}^d \int_0^t f_k^2(X_s)\,\mathrm{d}s\Big)$$

is a complex and bounded martingale. (Hint. Problem 1 with suitable λ and M).
Suggestion. If we compare the claim of Problem 2 and the Problem 1. we should try

$$M_t = \alpha \sum_{k=1}^d \int_0^t f_k(X_s)\,\mathrm{d}X_s^k$$

Then

$$\langle M, M \rangle_t = \alpha^2 \sum_{k,l=1}^{d} \int_0^t f_k(X_s) f_l(X_s) \, \mathrm{d}\langle X^k, X^l \rangle_s$$
$$= \alpha^2 \sum_{k=1}^{d} \int_0^t f_k^2(X_s) \, \mathrm{d}s$$

since $\langle X^k, X^l \rangle_s = s[\, k = l \,]$. With this choice the process Z^λ of the Problem 1 coincides with Y^f is

$$\lambda\alpha = i \quad \text{and} \quad -\lambda^2\alpha^2 = 1$$

This has many solutions, one of which is $\lambda = i$, $\alpha = 1$.

3. Suppose X is a $(\mathscr{F}_t)$-adapted continuous d-dimensional process, $X_0 = 0$ and let Y^f be the process as in Problem 2. Suppose for every $f = (f_1, \ldots, f_d)$ with $f_k \in L^2(\mathbb{R}^+)$ the process Y^f is complex and bounded $(\mathscr{F}_t)$-martingale. Show that

$$\mathbf{E}_0 \, [\, A \,] \exp\Big(i\xi \cdot (X_t - X_s) \Big) = \mathbf{P}_0 \, (\, A \,) \exp(-\tfrac{1}{2}|\xi|^2 (t-s))$$

holds for every $s < t < u$, every $\xi \in \mathbb{R}^d$ and every $A \in \mathscr{F}_s$. (Hint. $f = (f_1, \ldots, f_d)$ with $f_k(s) = \xi_k[\, s \le u \,]$)

Suggestion. The functions f_k that were given as a hint are in $L^2(\mathbb{R}_+)$ since

$$\int_0^\infty f_k^2(s) \, \mathrm{d}s = u \, \xi_k^2 < \infty$$

Therefore, the corresponding process Y^f is a complex and bounded martingale. Since Y^f is an exponential function, $Y_t^f \neq 0$ for every t and so $1/Y_s^f$ is well-defined. By the martingale property

$$\mathbf{E}_0 \Big([\, A \,] \, \frac{Y_t^f}{Y_s^f} \Big) = \mathbf{E}_0 \Big([\, A \,] \, \frac{\mathbf{E}_0 \big(Y_t^f \mid \mathscr{F}_s \big)}{Y_s^f} \Big) = \mathbf{E}_0 \Big([\, A \,] \, \frac{Y_s^f}{Y_s^f} \Big) = \mathbf{P}_0 \, (\, A \,)$$

holds for every $A \in \mathscr{F}_s$ and every $s < t < u$ and since

$$\frac{Y_t^f}{Y_s^f} = \exp\Big(i \sum_{k=1}^{d} \int_s^t f_k(X_v) \, \mathrm{d}X_v^k + \tfrac{1}{2} \sum_{k=1}^{d} \int_s^t f_k^2(X_v) \; \mathrm{d}v \Big)$$

we are almost done. The latter (ordinary) integral is

$$\sum_k \int_s^t f_k^2(X_v) \; \mathrm{d}v = \sum_k \xi_k^2 \int_s^t [\, v \le u \,] \; \mathrm{d}v = |\xi|^2 (t-s)$$

since $s < t < u$. The former integral with respect to X is

$$\sum_k \int_s^t f_k(X_v)\ \mathrm{d}X_v^k = \sum_k \xi_k \int_s^t [\, v \le u \,]\ \mathrm{d}X_v^k = \xi \cdot (X_t - X_s)$$

If we combine all these we obtain

$$\mathbf{E}_0\left([\, A\,]\ \frac{Y_t^f}{Y_s^f} \right) = \mathbf{E}_0\, [\, A\,] \exp(i\xi \cdot (X_t - X_s))\, e^{\frac{1}{2}|\xi|^2(t-s)} = \mathbf{P}_0\, (\, A\,)$$

which implies the claim.

4. Assume the same as in Problem 3. Show that for every $s < t$ the increment $X_t - X_s$ is independent from $\mathscr{F}_s$ and show X has the same expectation and variance as Brownian motion (i.e. show that X is $(\mathscr{F}_t)$-Brownian motion).
Suggestion. We know from Problem 3. that

$$\mathbf{E}_0\, [\, A\,] \exp(i\xi \cdot (X_t - X_s)) \ = \mathbf{P}_0\, (\, A\,)\ \ e^{-\frac{1}{2}|\xi|^2(t-s)}.$$

If we choose $A = \Omega$ and $s = 0$, then we obtain the characteristic function of X_t and we notice that it coincide with the characteristic function of the Brownian motion B_t at time t.

So it is enough to show the independence of the increment $X_t - X_s$ from $\mathscr{F}_s$. Let's denote the increment $Z = i(X_t - X_s)$ and $f_\xi(x) = \exp(x \cdot \xi)$. First we notice (again), then when $A = \Omega$ we have

$$\mathbf{E}_0\, f_\xi(Z) \ = e^{-\frac{1}{2}|\xi|^2(t-s)}$$

and therefore,

$$\mathbf{E}_0\, [\, A\,] f_\xi(Z) \ = \mathbf{P}_0\, (\, A\,)\, \mathbf{E}_0\, f_\xi(Z) \ = \mathbf{E}_0\, [\, A\,] \mathbf{E}_0\, f_\xi(Z)$$

for every $A \in \mathscr{F}_s$ and for every f_ξ. When $\mathbf{P}_0\, (\, A\,) \neq 0$, we can hence deduce that

$$\mathbf{E}_0(f_\xi(Z)\,|\,A) \ = \mathbf{E}_0\, f_\xi(Z)$$

for every ξ. If we define $\lambda(U) = \mathbf{P}_0\, (\, A\,)^{-1}\, \mathbf{P}_0\, (\, \{Z \in U\} \cap A\,)$ and $\mu(U) = \mathbf{P}_0\, (\, Z \in U\,)$ for every Borel set $U \subset \mathbb{R}^d$ we see that both λ and μ are Borel probability measures on $\mathbb{R}^d$ and moreover, that

$$\mathbf{E}_0(f(Z)\,|\,A) \ = \int_{\mathbb{R}^d} f(y) \lambda(\,\mathrm{d}y)$$

and

$$\mathbf{E}_0\, f(Z) \;=\; \int_{\mathbb{R}^d} f(y)\mu(\,\mathrm{d}y)$$

for every bounded and measurable function f (why? well, both can be easily seen to hold for simple functions and then monotone convergence gives the general case). Since f_ξ is a bounded and measurable function, we obtain that

$$\int_{\mathbb{R}^d} e^{i\xi\cdot y}\lambda(\,\mathrm{d}y) = \int_{\mathbb{R}^d} e^{i\xi\cdot y}\mu(\,\mathrm{d}y)$$

for every ξ or the characteristic functions of the probability measures λ and μ coincide. Thus, $\lambda = \mu$ which means that

$$\frac{\mathbf{P}_0\,(\,\{Z \in U\} \cap A\,)}{\mathbf{P}_0\,(\,A\,)} = \mathbf{P}_0\,(\,Z \in U\,)$$

for every Borel set U or

$$\mathbf{P}_0\,(\,\{Z \in U\} \cap A\,) = \mathbf{P}_0\,(\,A\,)\,\mathbf{P}_0\,(\,Z \in U\,)$$

for every Borel set U. This identity holds for every $A \in \mathscr{F}_s$ for which $\mathbf{P}_0\,(\,A\,) \neq 0$. However, the identity clearly holds also for every $\mathbf{P}_0\,(\,A\,) = 0$. Thus Z is independent from $\mathscr{F}_s$ and the claim follows.

In Problem 5-7 we look at convex functions.

5. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a convex function. Show that if $x' < x$ are two points on interval $(-r/2, r/2)$, then

$$f(x') - f(x) \leq |x - x'|\frac{f(-r) - f(x')}{r + x}.$$

(Hint: in this case $-r < x' < x$ and then convexity)
Suggestion. A function is convex if $f(x\theta + y(1-\theta)) \;\leq \theta f(x) + (1-\theta)f(y)$ for every $x < y$ and $\theta \in (0,1)$.

In this case x' is between $-r$ and x and so there exists $\theta \in (0,1)$ such that

$$x' = \theta x + (1-\theta)(-r) \quad \Longrightarrow (x+r)(1-\theta) = (x - x')$$

By convexity we have

$$f(x') \;\leq \theta f(x) + (1-\theta)f(-r)$$

or

$$f(x') - f(x) \le (1-\theta)(f(-r) - f(x)) = (x - x')\frac{f(-r) - f(x)}{x + r}$$

which is the *correct claim* since the claim contains a misprint.

6. Continuing with Problem 5. show that if f is convex and $|f| \le C$ on the interval $(-r, r)$ then

$$|f(x) - f(x')| \le \frac{4C}{r}|x - x'|$$

for every $x, x' \in (-r/2, r/2)$ or in other words, f is locally Lipschitz if f is locally bounded. (Hint: estimate the fraction in Problem 5, and repeat the construction for $x > x'$.)

Suggestion. According to Problem 5. we know that

$$f(x') - f(x) \le |x - x'|\frac{f(-r) - f(x)}{x + r} \le |x - x'|\frac{2C}{x + r}$$

Since $x > -r/2$, we have $x + r > r/2$ and so

$$f(x') - f(x) \le |x - x'|\frac{4C}{r}$$

If we change the roles of x and x' and we change $-r$ to r, we can repeat the Problem 5 and obtain

$$f(x) - f(x') \le |x - x'|\frac{f(r) - f(x')}{r - x'} \le |x - x'|\frac{2C}{r - x'}$$

Again, since $x' < r/2$, $r - x' > r/2$ and in both cases

$$f(x) - f(x') \le |x - x'|\frac{4C}{r}$$

This implies the claim.

7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex function. Let $\psi \in C^\infty$ which is zero outside an interval (a, b) for $a < b < 0$, which is positive and which integrates to 1. Define

$$f_n(x) = n\int_{-\infty}^{\infty} f(x + y)\psi(ny)\,\mathrm{d}y.$$

Show that f_n is convex. (Hint. use the definition of convexity directly).

Suggestion. Let us take $x < z$ and $\theta \in (0,1)$ and let's evaluate the f_n at the corresponding point and so

$$f_n(x\theta + z(1-\theta)) = n\int_{-\infty}^{\infty} f(x\theta + (1-\theta)z + y)\psi(ny)\,\mathrm{d}y.$$

Since $(1-\theta)y + \theta y = y$ for every y, we have

$$f(x\theta + (1-\theta)z + y) = f((x+y)\theta + (1-\theta)(z+y)) \leq \theta f(x+y) + (1-\theta)f(z+y)$$

Therefore,

$$\begin{aligned} f_n(x\theta + z(1-\theta)) &= n\int_{-\infty}^{\infty} f(x\theta + (1-\theta)z + y)\psi(ny)\,\mathrm{d}y \\ &\leq n\theta\int_{-\infty}^{\infty} f(x+y)\psi(ny)\,\mathrm{d}y \\ &+ n(1-\theta)\int_{-\infty}^{\infty} f(z+y)\psi(ny)\,\mathrm{d}y \\ &= \theta f_n(x) + (1-\theta)f_n(z) \end{aligned}$$

and the claim follows.

8. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex function and let ψ and f_n be as in Problem 7. Show that $f_n'(x) \to f_-'(x)$.
Suggestion. Using the change of variables $y' = x + y$, we have that

$$f_n(x) = n\int_{-\infty}^{\infty} f(y)\psi(n(y-x))\,\mathrm{d}y.$$

Since ψ is compactly supported and from lectures we know that f is locally bounded, so we can differentiate under the integral sign by dominated convergence theorem and thus,

$$f_n'(x) = -n^2\int_{-\infty}^{\infty} f(y)\psi'(n(y-x))\,\mathrm{d}y.$$

Let's change the integration variable once again, this time $y' = n(y-x)$, which means $y = x + hy'$ for $h = n^{-1}$ and $\mathrm{d}y' = n\,\mathrm{d}y$ and therefore,

$$f_n'(x) = -n\int_{-\infty}^{\infty} f(x+hy)\psi'(y)\,\mathrm{d}y.$$

Let's note that the left-handed derivative $f_-'(x)$ can be defined as

$$f_-'(x) = \lim_{h\to 0} \frac{f(x-|h|) - f(x)}{-|h|}$$

Since $f(x+hy)\psi'(y) = 0$ for every $y > b$ and $b < 0$, this means that if $f(x+hy)\psi'(y) \neq 0$, then $hy = -|hy|$. We can rewrite the expression we have for $f_n'(x)$ to accomodate this and so

$$\begin{aligned} f_n'(x) &= -n \int_{-\infty}^{\infty} f(x+hy)\psi'(y)\,\mathrm{d}y \\ &= -\int_{-\infty}^{\infty} \frac{f(x-|hy|)}{-|hy|} y\psi'(y)\,\mathrm{d}y \\ &= -\int_{-\infty}^{\infty} \frac{f(x-|hy|) - f(x) + f(x)}{-|hy|} y\psi'(y)\,\mathrm{d}y \end{aligned}$$

Assuming that we know that the left-handed derivative exists, then we can write

$$\frac{f(x-|hy|) - f(x) + f(x)}{-|hy|} y\psi'(y) = f_-'(x) y\psi'(y) + n f(x)\psi'(y) + \mathrm{o}(1) y\psi'(y)$$

If we assume the existence of the left-handed derivative at x the last error term vanish as $n \to \infty$. The second term seems problematic, since when $n \to \infty$ it explodes. However, the integral of this vanishes for every n and so the limit is zero. Let's verify this

$$-\int_{-\infty}^{\infty} n f(x)\psi'(y)\,\mathrm{d}y = -n f(x) \int_{2a}^{b/2} \psi'(y)\,\mathrm{d}y = -n f(x)(\psi(b/2) - \psi(2a)) = 0$$

since ψ vanishes outside the interval $(a, b) \subset (-\infty, 0)$. So by using the dominated convergence theorem and arguments above we have deduced that

$$\lim_{n\to\infty} f_n'(x) = -\int_{-\infty}^{\infty} f_-'(x) y\psi'(y)\,\mathrm{d}y = -f_-'(x)\Big(0 - \int_{-\infty}^{\infty} \psi(y)\,\mathrm{d}y\Big) = f_-'(x)$$

where we used the integration by parts, the fact that ψ vanish outside (a, b) and the fact that it integrates to one.

9. Suppose $t \mapsto p(t, x, y) \in C^1(\mathbb{R}_+)$ and $(x, y) \mapsto p(t, x, y) \in C^2(D) \cap C^1(\overline{D})$ and that p satisfies the heat equation

$$\begin{cases} \partial_t p(t,x,y) = \frac{1}{2}\triangle_x p(t,x,y), & \forall (t,x,y) \in (0,\infty) \times D \times D \\ \partial_\nu p(t,x,y) = 0 & \forall (t,x,y) \in (0,\infty) \times \partial D \times \overline{D} \\ p(0,x,\cdot) = \delta_x & \forall x \in \overline{D} \end{cases}$$

where $\partial_\nu p = \nu(x) \cdot \nabla_x p(t, x, y)$. Show that if p is unique, then

$$p(t+s, x, y) = \int_{\overline{D}} p(t, x, z) p(s, z, y)\,\mathrm{d}z$$

holds for every $t, s > 0$ and $x, y \in D$. (Hint. differentiate both sides with s and use the uniqueness of p)

Suggestion. The hint has it backwards, the differentation should be done with respect to t. First of all, if we denote $q(t,x) = p(t+s,x,y)$, then

$$\begin{cases} \partial_t q(t,x) = \frac{1}{2}\triangle_x q(t,x), & \forall (t,x) \in (0,\infty) \times D \\ \partial_\nu q(t,x) = 0 & \forall (t,x) \in (0,\infty) \times \partial D \\ q(0,x) = p(s,x,y) & \forall x \in \overline{D} \end{cases}$$

If we denote

$$\widetilde{q}(t,x) = \int_{\overline{D}} p(t,x,z)p(s,z,y)\,\mathrm{d}z$$

then

$$\widetilde{q}(0,x) = p(s,x,y)$$

and

$$\widetilde{q}(t,x) = \int_D p(t,x,z)p(s,z,y)\,\mathrm{d}z$$

when $t > 0$. Suppose $t > 0$. Since for every $x \in D$ the $z \mapsto \triangle_x p(t,x,z)$ is continuous and bounded function in $\overline{D}$ we have

$$\tfrac{1}{2}\triangle_x \widetilde{q}(t,x) = \int_D \tfrac{1}{2}\triangle_x p(t,x,z)p(s,z,y)\,\mathrm{d}z$$

for $x \in D$ and $t > 0$. Now since p satisfies the parabolic equation,

$$\tfrac{1}{2}\triangle_x \widetilde{q}(t,x) = \int_D \partial_t p(t,x,z)p(s,z,y)\,\mathrm{d}z$$

but again the derivative can be taken outside of the integral and we have

$$\tfrac{1}{2}\triangle_x \widetilde{q}(t,x) = \int_D \partial_t p(t,x,z)p(s,z,y)\,\mathrm{d}z = \partial_t \widetilde{q}(t,x)$$

for $t > 0$ and $x \in D$. In the same way we can move the differentiation to the normal direction inside the integration since we assumed that $x \mapsto p(t,x,y) \in C^1(\overline{D})$ and we get

$$\partial_\nu \widetilde{q}(t,x) = \int_D \partial_\nu p(t,x,z)p(s,z,y)\,\mathrm{d}z = 0$$

for every $x \in \partial D$ and $t > 0$. So we see that $\widetilde{q}$ also satisfies

$$\begin{cases} \partial_t \widetilde{q}(t,x) = \frac{1}{2}\triangle_x \widetilde{q}(t,x), & \forall (t,x) \in (0,\infty) \times D \\ \partial_\nu \widetilde{q}(t,x) = 0 & \forall (t,x) \in (0,\infty) \times \partial D \\ \widetilde{q}(0,x) = p(s,x,y) & \forall x \in \overline{D} \end{cases}$$

and so $u(t,x,y) = q(t,x) - \widetilde{q}(t,x)$ satisfies the homogenous equation

$$\begin{cases} \partial_t \widetilde{u}(t,x,y) = \frac{1}{2}\triangle_x \widetilde{q}(t,x,y), & \forall (t,x,y) \in (0,\infty) \times D \times D \\ \partial_\nu u(t,x) = 0 & \forall (t,x,y) \in (0,\infty) \times \partial D \times \overline{D} \\ \widetilde{u}(0,x,y) = 0 & \forall x \in \overline{D}, \forall y \in \overline{D} \end{cases}$$

By uniqueness of the equation that p solves we deduce that $u(t,x,y) = 0$ for $t \geq 0$, $x \in \overline{D}$ and $y \in \overline{D}$. But this implies the claim, since then $q(t,x) = \widetilde{q}(t,x)$ or

$$p(t+s,x,y) = \int_{\overline{D}} p(t,x,z)p(s,z,y)\,\mathrm{d}z.$$

10. Suppose p is as in Problem 9. Show that

$$\int_{\overline{D}} p(t,x,y)\,\mathrm{d}x = 1$$

for every $t \geq 0$ and $y \in \overline{D}$.
Suggestion. Let's denote by

$$m(t) = \int_D p(t,x,y)\,\mathrm{d}x$$

for $t > 0$ and let $y \in D$. Then $m'(t)$ satisfies

$$m'(t) = \int_D \partial_t p(t,x,y)\,\mathrm{d}x = \int_D \tfrac{1}{2}\triangle p(t,x,y)\ \,\mathrm{d}x$$

and by Gauß divergence theorem this is

$$\int_D \tfrac{1}{2}\triangle p(t,x,y)\ \,\mathrm{d}x = \tfrac{1}{2}\int_{\partial D} \partial_\nu p(t,x,y)\sigma(\,\mathrm{d}x) = 0.$$

Thus, $m(t) = C(y)$ is constant in time and therefore,

$$\int_D \int_D \varphi(y)p(t,x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_D \varphi(y)C(y)\,\mathrm{d}y$$

for every $t > 0$ and every function φ. Since the left-hand side converges to

$$\lim_{t\downarrow 0} \cdots = \int_D \varphi(y)\,\mathrm{d}y$$

we obtain

$$\int_D C(y)\varphi(y)\,\mathrm{d}y = \int_D \varphi(y)\,\mathrm{d}y$$

or

$$\int_D (C(y)-1)\varphi(y)\,\mathrm{d}y = 0$$

for every φ. If we choose $\varphi(y) = (C(y)-1)$, then

$$\int_D (C(y)-1)^2\,\mathrm{d}y = 0$$

or $C(y) = 1$ for almost every $y \in D$. Since

$$1 = C(y) = m(t) = \int_D p(t,x,y)\,\mathrm{d}x$$

for almost every y and the right-hand side is continuous in y, we obtain that

$$1 = \int_D p(t,x,y)\,\mathrm{d}x$$

for every $y \in \overline{D}$.

Note that there is a potential cheating in the suggestion above, since we didn't say clearly why we could change the order or integration and differentiation (since we could potentially have that $\triangle p$ is not integrable over D. A way to do this properly is to consider an increasing sequence of open sets $G_n \subset \overline{G_n} \subset D$ with smooth boundaries that become closer to the boundary of D as n grows. Then we would have

$$m(t) = \int_{G_n} p(t,x,y)\,\mathrm{d}x + \mathrm{o}(1) = m_n(t) + \mathrm{o}(1)$$

for $t < \rho$ for some fixed ρ. Now p is twice continuously differentiable in $\overline{G}_n$ and so we can justify the differentiation under the integral by dominated convergence. Thus, by similar method

$$m_n'(t) = \int_{G_n} \tfrac{1}{2}\triangle_x p(t,x,y)\,\mathrm{d}x = \tfrac{1}{2}\int_{\partial G_n} \partial_\nu p(t,x,y)\sigma(\,\mathrm{d}y).$$

Now since we assumed that p is continuously differentiable onto the boundary and $\partial_\nu p(t,x,y) = 0$ on ∂D we can deduce that for large n the derivative $|m_n'(t)| < \varepsilon$ for $t < \rho$. Therefore,

$$|m_n(t) - m_n(0)| \le \varepsilon\rho$$

and combining this with $m(t) = m_n(t) + \mathrm{o}(1)$ we can say that

$$m(t) = m_n(0) + \mathrm{o}(1)$$

for $t < \rho$. In particular, if we choose $0 < s < \rho$, we obtain

$$m(t) - m(s) = \mathrm{o}(1)$$

and now letting $n \to \infty$, we get $m(t) = m(s)$ and we can proceed as before. Note also that we didn't just write $m(0) = 1$ since we didn't have a meaning for integral like

$$\int_{\overline{D}} \delta_x(\cdot)\,\mathrm{d}x$$

but we could interpret it via

$$\int_{\overline{D}} \varphi(y) \int_{\overline{D}} p(t,x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\overline{D}} \int_{\overline{D}} \varphi(y)p(t,x,y)\,\mathrm{d}y\,\mathrm{d}x$$

and then using the compactness on $\overline{D}$ and the convergence

$$\lim_{t\to 0} \int_{\overline{D}} \varphi(y)p(t,x,y)\,\mathrm{d}y = \varphi(x)$$

for every $x \in \overline{D}$. Moreover, we need φ to be continuous, and therefore, the choice $\varphi(y) = C(y) - 1$ is allowed only when C is continuous, but this follows from the continuity of p with respect to y.

11. Suppose p is as in Problem 9 and assume that p is unique. If we know in addition that $p(t,x,y) \geq 0$ for $t \geq 0$, show that p is a probability transition density of some Feller process (X_t) with the state space $\overline{D}$.

Suggestion. So we need to show that the transition function P_t of X is given by

$$P_t f(x) = \int_{\overline{D}} p(t,x,y)f(y)\,\mathrm{d}y.$$

For this we only need to verify that (P_t) is a Feller semigroup. From the assumption $p \geq 0$, it follows that $P_t f \leq P_t g$ for every $f \leq g$, since if $g \geq f$, then

$$P_t(g-f)(x) = \int_{\overline{D}} p(t,x,y)(g-f)(y)\,\mathrm{d}y \geq 0.$$

The fact that $P_t 1 = 1$ can be shown as follows. Since

$$P_t 1(x) = \int_{\overline{D}} p(t,x,y)\,\mathrm{d}y$$

we notice that $f = P_t 1$ solves the equation

$$\begin{cases} \partial_t f = \frac{1}{2}\triangle f \\ \partial_\nu f = 0 \end{cases}$$

by differentiation in the same way as in Problem 9. When $t = 0$, we have that $P_0 1(x) = 1$ for every $x \in \overline{D}$. The function $v(t, x) = 1$ solves the same equation with the same initial value $x \mapsto v(0, x) = 1$. By uniqueness, $f = v$, or $P_t 1(x) = 1$ for every x.

The semigroup property follows from Problem 9, since

$$\begin{aligned} P_{t+s} f(x) &= \int_{\overline{D}} p(t + s, x, y) f(y) \, \mathrm{d}y = \int_{\overline{D}} \int_{\overline{D}} p(t, x, z) p(s, z, y) f(y) \, \mathrm{d}z \, \mathrm{d}y \\ &= \int_{\overline{D}} p(t, x, z) P_s f(z) \, \mathrm{d}z = P_t P_s f(x) \end{aligned}$$

and the fact that $\lim_{t \downarrow 0} P_t f(x) = f(x)$ is the initial value assumption for the function p. The only thing to verify still is that $P_t f \in C_\infty(\overline{D})$ when $f \in C_\infty(\overline{D})$. Since $\overline{D}$ is compact, this is equivalent with $P_t f \in C(\overline{D})$ when $f \in C(\overline{D})$. But this follows since $x \mapsto p(t, x, y)$ is continuous function and so $P_t f \in C(\overline{D})$ even for every bounded and measurable $f \colon \overline{D} \to \mathbb{R}$.