Department of Mathematics and Statistics
Stochastic processes on domains
Suggestions to excercise problem sheet 5

Note. In the Problems 1-12 the j,k and n are always integers.

1. Suppose M and N are bounded martingales and M and N are independent. Show that

$$\langle M, N \rangle = 0.$$

Suggestion. We should also assume that M and/or N is continuous. So let's suppose that both of them is continuous. Now

$$\langle M, N \rangle = \frac{1}{4}(\langle M+N, M+N \rangle - \langle M-N, M-N \rangle)$$

Since we know form lectures that

$$\langle M \pm N, M \pm N \rangle_t = \lim_{(t_1,\dots,t_n)} \sum_k (M_{t_{k+1}} \pm N_{t_{k+1}}) - (M_{t_k} \pm N_{t_k}))^2$$

in probability where the $0 = t_0 < t_1 < \cdots < t_n = t$ is some division such that limit means that $\sup_k |t_{k+1} - t_k| \to 0$, we can deduce that

$$\langle M, N \rangle_t = \lim_{(t_1,\dots,t_n)} \sum_k (M_{t_{k+1}} - M_{t_k})(N_{t_{k+1}} - N_{t_k}).$$

So if we can show that the sum on the right-hand side goes to zero in the limit we are done. Let's denote $\widetilde{M}_k := (M_{t_{k+1}} - M_{t_k})$ and $\widetilde{N}_k := (N_{t_{k+1}} - N_{t_k})$ and thus the claim is to show that

$$\lim_{(t_1,\dots,t_n)} \mathbf{P}\left(\sum_k \widetilde{M}_k \widetilde{N}_k > \varepsilon \right) = 0$$

for every $\varepsilon > 0$. One way to show this is to show that $\lim \mathbf{E}\,(\sum \dots)^2 = 0$, since then the claim follows by Chebysev's inequality. So let's try to compute the second moment of the sum

$$\mathbf{E}\left(\sum_k \widetilde{M}_k \widetilde{N}_k \right)^2 = \mathbf{E} \sum_{k,l} \widetilde{M}_k \widetilde{M}_l \widetilde{N}_k \widetilde{N}_l = \sum_{k,l} \mathbf{E}\,(\widetilde{M}_k \widetilde{M}_l)\, \mathbf{E}\,(\widetilde{N}_k \widetilde{N}_l)$$

where we used the independence of M and N. If we have $k > l$, then

$$\mathbf{E}\, \widetilde{M}_k \widetilde{M}_l = \mathbf{E}\, \widetilde{M}_l \mathbf{E}\left(\widetilde{M}_l \,|\, \mathscr{F}_{t_l} \right) = 0$$

since M is a martingale. The same is true, when $k < l$, and for terms with N as well, so

$$\mathbf{E}\left(\sum_k \widetilde{M}_k \widetilde{N}_k\right)^2 = \mathbf{E}\sum_{k,l} \widetilde{M}_k \widetilde{M}_l \widetilde{N}_k \widetilde{N}_l = \sum_k \mathbf{E}\,\widetilde{M}_k^2\,\mathbf{E}\,\widetilde{N}_k^2\,.$$

Since

$$\mathbf{E}\,\widetilde{M}_k^2 = \mathbf{E}\,M_{t_{k+1}}^2 + \mathbf{E}\,M_{t_k}^2 - 2\mathbf{E}\,M_{t_k} M_{t_{k+1}}$$

and since M is a martingale

$$\mathbf{E}\,M_{t_k} M_{t_{k+1}} = \mathbf{E}\,M_{t_k}^2 + \mathbf{E}\,M_{t_k}\mathbf{E}\left(M_{t_{k+1}} - M_{t_k} \mid \mathscr{F}_{t_k}\right) = \mathbf{E}\,M_{t_k}^2$$

we have that

$$\mathbf{E}\,\widetilde{M}_k^2 = \mathbf{E}\,M_{t_{k+1}}^2 - \mathbf{E}\,M_{t_k}^2$$

and the same holds for N. We use a trivial estimate $\mathbf{E}\,\widetilde{N}_k^2 \le \max_k \mathbf{E}\,\widetilde{N}_k^2 =: \rho$ and thus

$$\sum_k \mathbf{E}\,\widetilde{M}_k^2\,\mathbf{E}\,\widetilde{N}_k^2 \le \rho \sum_k \mathbf{E}\,\widetilde{M}_k^2 = \rho(\mathbf{E}\,M_{t_n}^2 - \mathbf{E}\,M_0^2)$$

where we could compute the sum, since it was telescoping sum. Since M is bounded martingale we have that $\mathbf{E}\,M_t^2 \le K < \infty$ and so

$$\sum_k \mathbf{E}\,\widetilde{M}_k^2\,\mathbf{E}\,\widetilde{N}_k^2 \le \rho K = K \max_k \mathbf{E}\,\widetilde{N}_k^2 = K \max_k(\mathbf{E}\,N_{t_{k+1}}^2 - \mathbf{E}\,N_{t_k}^2)$$

We are almost done, since we assumed that N is bounded and continuous and thus, the function $s \mapsto \mathbf{E}\,N_s^2$ is continuous. On a bounded interval $[0, t]$ it is even uniformly continuous and thus

$$\max_k(\mathbf{E}\,N_{t_{k+1}}^2 - \mathbf{E}\,N_{t_k}^2) \le \varepsilon/K$$

whenever the the maximum distance $\max_k|t_{k+1} - t_k|$ is small enough. All in all,

$$\mathbf{E}\left(\sum_k \widetilde{M}_k \widetilde{N}_k\right)^2 \le \varepsilon$$

whenever the the maximum distance $\max_k|t_{k+1} - t_k|$ is small enough and especially,

$$\limsup_{(t_1,\ldots,t_n)} \mathbf{E}\left(\sum_k \widetilde{M}_k \widetilde{N}_k\right)^2 \le \varepsilon$$

holds for every $\varepsilon > 0$. This implies the claim.

2. Suppose f is a continuous function on the boundary of a ball $D_r(x)$ and η is the first exist time from the ball $D_r(x)$. Show that

$$\mathbf{E}_x\, f(B_\eta) = \int f(y)\mu(\,\mathrm{d}y)$$

where μ is the normalised surface measure of the sphere $\partial D_r(x)$. (Hint: if f is an indicator function, show that the μ has to rotation invariant. You may assume that you know that then it must be the surface measure)

Suggestion. Let's denote the boundary $\partial D_r(x)$ by Γ, i.e. $\Gamma = \partial D_r(x)$ and the ball $D_r(x)$ by G. let's denote the mapping that the left-hand side induces by Λ, i.e. $\Lambda = f \mapsto \mathbf{E}_x\, f(B_\eta)$.

When $f = [\,A\,]$ for some set $A \subset \Gamma$, we will denote

$$\lambda(A) = \Lambda([\,A\,]).$$

When $f(y) = [\,y \in \Gamma\,]$, then the left-hand side is one, i.e. $\lambda(\Gamma) = 1$.

If $(A_k) \subset \mathscr{B}(\Gamma)$ are Borel sets on Γ and $A_k \cap A_j = \emptyset$ for $k \neq j$, we notice that

$$\Lambda([\bigcup_k A_k\,]) = \sum_{k=1}^{\infty} \Lambda[\,A_k\,]$$

i.e.

$$\lambda(\bigcup_k A_k) = \sum_{k=1}^{\infty} \lambda(A_k)$$

and thus, λ is a Borel probability measure on Γ. By linearity of Λ we first deduce for simple functions $f = \sum_k a_k[\,A_k\,]$ that

$$\Lambda f = \int_\Gamma f(y)\lambda(\,\mathrm{d}y).$$

Moreover, by monotone convergence we may deduce that

$$\mathbf{E}_x\, f(B_\eta) = \Lambda f = \int_\Gamma f(y)\lambda(\,\mathrm{d}y)$$

for every bounded Borel measurable f on Γ. Therefore, the claim follows if we can show that $\lambda = \mu$ which follows by showing that λ is rotation invariant, i.e. $\lambda(RA) = \lambda(A)$ for every rotation R around x.

Now $\lambda(RA) = \mathbf{E}_x\,[\,B_\eta \in RA\,] = \mathbf{P}_x\,(\,R^{-1}B_\eta \in A\,)$. Since $X_t = R^{-1}B_t$ is a Brownian motion starting from x whenever B is a Brownian motion starting from x and if η' is the first exit time of X from the ball G, then by rotation symmetry of Brownian motion we notice that (η, B) has the same law than (η', X). Therefore,

$$\mathbf{P}_x\left(\,R^{-1}B_\eta \in A\,\right) = \mathbf{P}_x\,(\,X_{\eta'} \in A\,) = \mathbf{P}_x\,(\,B_\eta \in A\,)$$

or $\lambda(RA) = \lambda(A)$.

3. Suppose $w(x) = \mathbf{E}_x\, w(B_\eta)$ as in the proof of Lemma 7.5. Show that

$$w(x) = \int_{D_r(x)} w(y)\varphi(|y-x|)\,\mathrm{d}y$$

for every $\varphi\colon \mathbb{R} \to \mathbb{R}^+$ such that $\int \varphi(t)\,\mathrm{d}t = 1$ and $\varphi(t) = 0$ outside interval $(r/2, r)$. (Hint. use Problem 2. and Fubini to the right-hand side to separate w and φ.)

Suggestion. First: the claim of this is problem is not entirely correct, so you'll get this for free. But let's show this and correct the claim in the end. By problem 2.

$$w(x)\varphi(t) = \int_{\partial D_t(x)} \varphi(t) w(y)\mu_t(\,\mathrm{d}y)$$

for every $t > 0$ small enough. We can integrate this identity over t from 0 to r and we obtain

$$w(x)\int_0^r \varphi(t)\,\mathrm{d}t = \int_0^r \int_{\partial D_t(x)} \varphi(t) w(y)\mu_t(\,\mathrm{d}y)\,\mathrm{d}t$$

The integral on the left hand side is 1 so the left-hand side is $w(x)$. The integral on the right for fixed $t > 0$ is

$$\int_{\partial D_t(x)} \varphi(|y-x|) w(y) t^{-(d-1)}\ \mu_t(\,\mathrm{d}y)$$

The measure μ_t is the normalised surface measure of the sphere of radius t, so

$$\mu_t(\partial D_t) = \sigma(\partial D_t) t^{-(d-1)}$$

where σ is the surface measure. By Fubini's theorem and representation with polar coordinates we know that

$$\int_0^r \int_{\partial D_t(x)} f(t,y)\sigma(\,\mathrm{d}y)\,\mathrm{d}t = \int_{D_r(x)} f(|y-x|, y)\,\mathrm{d}y$$

for every f so

$$\int_0^r \int_{\partial D_t(x)} f(t,y)\mu_t(\,\mathrm{d}y)\,\mathrm{d}t = \int_{D_r(x)} f(|x-y|, y)|y-x|^{-(d-1)}\,\mathrm{d}y.$$

Combining these we obtain that

$$w(x) = \int_{D_r(x)} \varphi(|x-y|)|x-y|^{-(d-1)} w(y)\,\mathrm{d}y = \int_{D_r(x)} \psi(|x-y|) w(y)\,\mathrm{d}y$$

which is *the correct claim.* Notice that since $\varphi(|x-y|) = 0$ for every $|y-x| < r/2$, the $\psi(t) = \varphi(t)t^{-(d-1)}$ is C^∞-function as well.

In other words, the claim should be corrected so that

$$\int_0^\infty t^{d-1}\varphi(t)\,\mathrm{d}t = 1.$$

4. Suppose $w(x) = \mathbf{E}_x\, w(B_\eta)$ as in the proof of Lemma 7.5. Show that w is $C^\infty(G)$. (Hint. use previous problem 3. and differentiate. You may assume the existence of C^∞ functions that vanish outside $(r/2, r)$.

Suggestion. Let $\varphi\colon \mathbb{R}^+ \to \mathbb{R}$ be a C^∞-function that vanish outside the interval $(r/2, r)$ and that $\int \varphi(t)t^{d-1}\,\mathrm{d}t = 1$. By Problem 3. we know that

$$w(x) = \int_{\mathbb{R}^d} \varphi(|x-y|)w(y)\,\mathrm{d}y. \qquad (*)$$

According to Lemma 7.5. we know that w is bounded, so using the dominated convergence theorem to identity $(*)$, we can deduce that w is continuous, since

$$w(x_n) = \int_{\mathbb{R}^d} \varphi(|x_n-y|)w(y)\,\mathrm{d}y \to \int_{\mathbb{R}^d} \varphi(|x-y|)w(y)\,\mathrm{d}y = w(x).$$

In the same way, we can compute the first order differences

$$h^{-1}(w(x+he_j) - w(x)) = \int_{\mathbb{R}^d} h^{-1}(\varphi(|x+he_j-y|) - \varphi(|x-y|))w(y)\,\mathrm{d}y.$$

Now

$$\varphi(|x+he_j-y|) - \varphi(|x-y|) = \rho(h) - \rho(0) = h\rho'(0) + \mathrm{o}(h)$$

when $\rho(t) = \varphi(|x-y+te_j|)$. Since

$$\rho'(0) = \varphi'(|x-y|)(x_j-y_j)|x-y|^{-1}$$

we see that $\varphi_j(x,y) := \varphi'(|x-y|)(x_j-y_j)|x-y|^{-1}$ is a continuous function that vanish when $|x-y| < r/2$ or $|x-y| > r$. Thus we may again use the dominated convergence theorem and we obtain that

$$\partial_j w(x) = \int_{\mathbb{R}^d} \varphi_j(x,y)w(y)\,\mathrm{d}y$$

and therfore, $w \in C^1(G)$. We can repeat this as many times as we wish and we obtain that

$$\partial_{j_1\ldots j_N} w(x) = \int_{\mathbb{R}^d} \varphi_{j_1,\ldots,j_N}(x,y)w(y)\,\mathrm{d}y$$

for some C^∞-function $\varphi_{j_1,\dots,j_N}$ that is supported on $r/2 < |x-y| < r$. This implies the claim.

5. Show that the for every $z \in G$

$$\mathbf{P}_z\left(\tau \le t\right) = \lim_{n\to\infty} \mathbf{E}_z\,\mathbf{P}_{B(n^{-1})}\left(\tau \le t - n^{-1}\right)$$

and that $z \mapsto \mathbf{E}_z\,\mathbf{P}_{B(n^{-1})}\left(\tau \le t - n^{-1}\right)$ is continuous (even C^∞) for every n. (Hint. Markov property and the transition probability density.)

Suggestion. Let's start with the identity

$$A := \{\tau \le t\} = \{\exists s \in (0,t] : B(s) \notin G\}.$$

The reason that this holds is that if $\tau > t$, then $B(s) \in G$ for every $s \in (0,t]$. On the other hand, if $B(s) \in G$ for every $s \in (0,t]$ then since G is open and B is continuous, there exists a $s' > t$ such that $B(u) \in G$ for every $u \in (0,s')$ and so $\tau \ge s' > t$.

This identity can be expanded written as a union of increasing events, namely

$$A = \bigcup_{n=1}^{\infty} A_n := \bigcup_{n=1}^{\infty} \{\exists s \in (n^{-1},t] : B(s) \notin G\}.$$

Therefore,

$$\mathbf{P}_z\left(A\right) = \lim_{n\to\infty} \mathbf{P}_z\left(A_n\right) = \lim_{n\to\infty} \mathbf{E}_z\,\mathbf{P}_z\left(A_n \,|\, \mathscr{F}_h\right)$$

where $h = n^{-1}$. The conditional probability on the right is by Markov property

$$\mathbf{P}_z\left(A_n \,|\, \mathscr{F}_h\right) = \mathbf{P}_{B_h}\left(\exists s \in (0,t-h]\colon B_s \notin G\right) = \mathbf{P}_{B_h}\left(\tau \le t-h\right)$$

Therefore,

$$\mathbf{P}_z\left(\tau \le t\right) = \lim_{n\to\infty} \mathbf{E}_z\,\mathbf{P}_{B_h}\left(\tau \le t-h\right)$$

which gives the first claim.

Let's denote $A_n = \{\tau \le t-h\}$. The second claim is to show that $f_n = z \mapsto \mathbf{E}_z\,\mathbf{P}_{B_h}\left(A_n\right)$ is a continuous function (even in $\mathbb{R}^d$). Now since $g_n = y \mapsto \mathbf{P}_y\left(A_n\right)$ is measurable and bounded by 1, we can rewrite the function f_n as $f_n(z) = \mathbf{E}_z\,g_n(B_h)$ and since $h > 0$ we can express this with the help of the transition probability density of Brownian motion, i.e.

$$f_n(z) = \mathbf{E}_z\,g_n(B_h) = \int_{\mathbb{R}^d} g_n(y)p(h,z,y)\,\mathrm{d}y.$$

If $z_k \to z$, then

$$|f_n(z) - f_n(z_k)| \leq \int_{\mathbb{R}^d} |p(h,z,y) - p(h,z_k,y)| \, \mathrm{d}y.$$

THe right-hand side has an integrable majorant and since since $z \mapsto p(h,z,y)$ is continuous for every $y \in \mathbb{R}^d$ we deduce with dominated convergence theorem that f_n is continuous.

Note that it was important to move the Brownian motion first a bit to B_h since even though $\mathbf{P}_z(\tau \leq t) = \mathbf{E}_z \mathbf{P}_{B_0}(\tau \leq t) = \mathbf{E}_z g_\infty(B_0)$ formally in the same way, the latter expression cannot be written with the help of transition probability density, since at time $t = 0$ the Brownian motion does not have one.

Moreover, this does not mean that $z \mapsto \mathbf{P}_z(\tau \leq t)$ would be continuous.

6. Show that the for every $x \in \partial G$ and every $(x_n) \subset G$ such that $x_n \to x$ it holds that

$$\mathbf{P}_x(\tau \leq t) \leq \liminf_{n\to\infty} \mathbf{P}_{x_n}(\tau \leq t)$$

(Hint: use Problem 5 to deduce this lower semicontinuity property by approximating from below by continuous functions)

Suggestion. Let's denote $f_n(z) = \mathbf{E}_z \mathbf{P}_{B_h}(A_n)$ as in the suggestion for the Problem 5 and let (x_k) and x be as in the claim of the problem. Moreover, let's denote $f(z) = \mathbf{P}_z(\tau \leq t)$. We showed that for every x_k it holds that

$$f(x_k) = \lim_{n\to\infty} f_n(x_k)$$

and that f_n are continuous functions. In addition we showed that $f_n(z) \uparrow f(z)$ for every z. Therefore,

$$\liminf_{k\to\infty} f(x_k) \geq \liminf_{k\to\infty} \; f_n(x_k) = f_n(x)$$

since f_n is continuous and $x_k \to x$. Moreover, since $f_n \uparrow f$, the definition of the supremem implies that

$$\liminf_{k\to\infty} f(x_k) \geq \sup_n f_n(x) = f(x)$$

and the claim follows.

7. Show that if x is a regular point on the boundary and $(x_n) \subset G$ such that $x_n \to x$, then

$$\mathbf{P}_{x_n}(\tau \leq t) = 1$$

for every $t > 0$. (Hint. Problem 6.)
Suggestion. First the claim is missing the limit and the real claim is

$$\lim_{n\to\infty} \mathbf{P}_{x_n}(\tau \le t) = 1.$$

For this we only need to apply Problem 6 which says that

$$1 \ge \limsup_{n\to\infty} \mathbf{P}_{x_n}(\tau \le t) \ge \liminf_{n\to\infty} \mathbf{P}_{x_n}(\tau \le t) \ge \mathbf{P}_x(\tau \le t) \ge \mathbf{P}_x(\tau = 0) = 1$$

by the regularity of the point $x \in \partial G$. This implies that

$$1 = \liminf \mathbf{P}_{x_n}(\tau \le t) = \limsup \mathbf{P}_{x_n}(\tau \le t)$$

which shows that the limit exists and is 1.

8. Show that 0 is a regular point of $(0, 1)$ for 1-dimensional Brownian motion without using flat cone condition. (Hint. Blumenthal 0-1 -law).
Suggestion. First we notice that

$$\mathbf{P}_0(\tau = 0) = \lim_{t\downarrow 0} \mathbf{P}_0(\tau \le t)$$

Since $\mathbf{P}_0(\tau \le t) = \mathbf{P}_0(B_t \le 0) + \mathbf{P}_0(B_t > 0, \tau < \tau) \ge \mathbf{P}_0(B_t \le 0) = \frac{1}{2}$, we have that

$$\mathbf{P}_0(\tau = 0) \ge \tfrac{1}{2}$$

By Blumenthal 0-1 -law, this implies that $\mathbf{P}_0(\tau = 0) = 1$ which implies the claim.

9. Prove the Blumenthal's 0-1 -law. i.e. show that when $\mathscr{F}_0$ is augmented history of Brownian motion, then if $A \in \mathscr{F}_{0^+} = \mathscr{F}_0$, we either have $\mathbf{P}_x(A) = 0$ or $\mathbf{P}_x(A) = 1$. (Hint. consider the random variable $[A][A]$ and use Markov property to deduce that $\mathbf{E}_x[A][A] = \mathbf{P}_x(A)^2$.)
Suggestion. Let's follow the hint and compute $\mathbf{E}_x[A][A]$ for $A \in \mathscr{F}_0$ in two ways. Since $[A][A] = [A]$, we have that $\mathbf{E}_x[A][A] = \mathbf{P}_x(A)$. On the other hand, the Markov property and the $\mathscr{F}_0$-measurability of $[A]$ imply that

$$\begin{aligned}\mathbf{E}_x[A][A] &= \mathbf{E}_x[A]\mathbf{P}_x(A \mid \mathscr{F}_0) = \mathbf{E}_x[A]\mathbf{P}_{B_0}(A) = \mathbf{E}_x[A]\mathbf{P}_x(A)\\ &= (\mathbf{P}_x(A))^2\end{aligned}$$

Therefore, we obtain an equation $\mathbf{P}_x(A) = \mathbf{P}_x(A)^2$ for every $A \in \mathscr{F}_0$. This second order polynomial equation $\alpha = \alpha^2$ only has two solutions $\alpha = 0$ and $\alpha = 1$ and the claim follows.

10. Suppose $\frac{1}{2}\triangle u = g$ in domain G. If u is $C^2(G)$ and g is bounded, show that

$$Z_t = u(B_t) - \int_0^t g(B_s)\,\mathrm{d}s$$

is a continuous local martingale in $[0, \tau)$ for every starting point x.

Suggestion. The tool we have is Itō, so let's use it to $X_t = u(B_t)$. Since u is in $C^2(G)$, then $X_t^{\tau'}$ is in G for every stopping time $\tau' < \tau$ and the following is well defined

$$X_t^{\tau'} = X_0 + \int_0^{t\wedge\tau'} \nabla u(B_s)\cdot \mathrm{d}B_s + \tfrac{1}{2}\int_0^{t\wedge\tau'} \triangle u(B_s)\,\mathrm{d}s.$$

Since $\frac{1}{2}\triangle u(B_s) = g(B_s)$ for every $s \le \tau'$ we obtain that

$$Z_t^{\tau'} = X_t^{\tau'} - \int_0^{t\wedge\tau'} g(B_s)\,\mathrm{d}s = u(B_0) + \int_0^{t\wedge\tau'} \nabla u(B_s)\cdot \mathrm{d}B_s$$

Now let $G_n \subset G$ be an open set such that $G_n \subset G_{n+1}$ and $\overline{G}_n \subset G$ and $\bigcup G_n = G$. Let τ_n be the first exit time from G_n.

Let $x \in G$ and so there is an N such that $x \in G_N$. By the continuity of the Brownian motion, we observe that $\tau_n \uparrow \tau$ for $\mathbf{P}_x$-almost surely. Moreover, $\tau_n < \tau$ for $n > N$ for $\mathbf{P}_x$-almost surely.

Why we introduced these sets G_n? Namely, because $u \in C^2(G)$ the ∇u is bounded in $\overline{G}_n$ for every n. Note that we didn't assume this from the beginning. This implies now that

$$Z_t^{\tau_n} = u(x) + \int_0^{t\wedge\tau_n} \nabla u(B_s)\cdot \mathrm{d}B_s$$

is a bounded martingale for starting point x. Therefore, Z is a continuous local martingale on $[0, \tau)$.

11. Suppose $\frac{1}{2}\triangle u = qu$ in domain G. If u is $C^2(G)$ and $q \le 0$, show that

$$Z_t = u(B_t)e^{-\int_0^t q(B_s)\,\mathrm{d}s}$$

is a continuous local martingale in $[0, \tau)$ for every starting point x.

Suggestion. A misprint again, since we want $q \ge 0$. Let $f(x, y) = u(x)e^{-y}$ and define

$$A_t = \int_0^t q(B_s)\,\mathrm{d}s$$

This is an increasing process, since $q \geq 0$ on the interval $[0, \tau)$. Moreover, $\mathrm{d}A_t = q(B_t)\,\mathrm{d}t$. Now if we apply Itō to $Z_t = f(B_t, A_t)$, we get that

$$Z_t = Z_0 + \int_0^t \nabla u(B_s) e^{-A_s}\,\mathrm{d}B_s - \int_0^t u(B_s) q(B_s)\,\mathrm{d}s + \tfrac{1}{2} \int_0^t \triangle u(B_s)\,\mathrm{d}s$$

for every $t < \tau$. As in Problem 10. we could make this more rigorous by looking at G_n's and τ_n's so that we can be sure that all the integrals on the right are well defined. But all in all, when $s < t < \tau$, the term $-u(B_s)q(B_s) = \frac{1}{2}\triangle u(B_s)$ vanishes by assumption and the only thing that is left is a stochastic integral with respect to a continuous local martingale on $[0, \tau)$. Thus, the claim follows.

12. Suppose $\frac{1}{2}\triangle u = g$ in domain G and $u = f$ on ∂G. If u is $C^2(G)$ and it is continuous in $\overline{G}$, and f is bounded, show that

$$u(x) = \mathbf{E}_x\, f(B_\tau) - \mathbf{E}_x \int_0^\tau g(B_s)\,\mathrm{d}s$$

Suggestion. It should be added that G is bounded as we have had throughout the lectures. Then $\mathbf{E}_x\, \tau < \infty$ as we have seen before. Borrowing from the suggestion for Problem 10. we have that Z^{τ_n} is a bounded martingale. Therefore,

$$\mathbf{E}_x\, Z^{\tau_n}_{\tau_n} = \mathbf{E}_x\, Z^{\tau_n}_0$$

for every $n > N$ when $x \in G_N$. This means that

$$\mathbf{E}_x\, Z_0 = \mathbf{E}_x\, u(B_0) = u(x)$$

is equal to

$$\mathbf{E}_x\, Z^{\tau_n}_{\tau_n} = \mathbf{E}_x\, u(B_{\tau_n}) - \mathbf{E}_x \int_0^{\tau_n} g(B_s)\,\mathrm{d}s$$

for every $n > N$. This implies that

$$u(x) = \lim_{n\to\infty} \Big(\mathbf{E}_x\, u(B_{\tau_n}) - \mathbf{E}_x \int_0^{\tau_n} g(B_s)\,\mathrm{d}s \Big)$$

Since u is continuous in $\overline{G}$ and B is continuous and $\tau_n \uparrow \tau$, the first limit is

$$\lim_{n\to\infty} \mathbf{E}_x\, u(B_{\tau_n}) = \mathbf{E}_x\, u(B_\tau) = \mathbf{E}_x\, f(B_\tau)$$

by the dominated convergence theorem and the fact that $u = f$ on the boundary. Moreover, since

$$|[\, s \leq \tau_n\,]\, g(B_s)| \leq [\, s \leq \tau\,] \|\, g\,\|_\infty$$

and

$$\mathbf{E}_x \int {}_0^\infty [\, s \le \tau \,] \|\, g \,\|_\infty = \|\, g \,\|_\infty \mathbf{E}_x \, \tau \; < \infty$$

we may again use dominated convergence and

$$\lim_{n\to\infty} \mathbf{E}_x \int_0^{\tau_n} g(B_s)\,\mathrm{d}s \; = \mathbf{E}_x \int_0^\infty \lim_{n\to\infty} [\, s \le \tau_n \,] g(B_s)\,\mathrm{d}s \,.$$

Since $\tau_n < \tau$ and $\tau_n \uparrow \tau$, we can compute this limit exactly and we obtain

$$\lim_{n\to\infty} [\, s \le \tau_n \,] \; = [\, s < \tau \,]$$

and this gives the claim.