Department of Mathematics and Statistics
Stochastic processes on domains
Suggestions to excercise problem sheet 4

1. Suppose M and N are bounded martingales. Show that

$$\langle M, N \rangle^\tau = \langle M^\tau, N \rangle.$$

(Hint: use uniqueness in Theorem 6.10. and Optional Stopping Theorem.)
Suggestion. Since $MN - \langle M, N \rangle$ is a continuous local martinale and since it is bounded, it is a martingale (see Problem 9, which implies that every bounded continuous local martingale is a martingale).

Therefore, we have by Optional Stopping Theorem, that

$$\bar{M} := M^\tau N^\tau - \langle M, N \rangle^\tau$$

is a bounded martingale. We also have that

$$M^\tau N - \langle M^\tau, N \rangle$$

is a continuous bounded martingale and thus, if we can show that

$$\bar{N} := M^\tau (N - N^\tau)$$

is a bounded martingale, then

$$M^\tau N - \langle M, N \rangle^\tau = \bar{N} + \bar{M}$$

is also a bounded martingale. Therefore, the uniqueness implies the claim, once we have verified that $\bar{N} := M^\tau (N - N^\tau)$ is a martingale.

In order to show this, let η be bounded stopping time. Now

$$\bar{N}_\eta = M_{\tau\wedge\eta}(N_\eta - N_{\tau\wedge\eta}).$$

If $\eta \le \tau$, then $N_{\tau\wedge\eta} = N_\eta$, so the right-hand side vanishes. In other words,

$$\bar{N}_\eta = [\,\eta > \tau\,] M_\tau (N_\eta - N_\tau).$$

Therefore, by taking the conditional expectation at the stopping time τ, we obtain

$$\mathbf{E}\left(\bar{N}_\eta \,|\, \mathscr{F}_\tau\right) = [\,\eta > \tau\,] M_\tau (\mathbf{E}\,(N_\eta \,|\, \mathscr{F}_\tau) - N_\tau).$$

By Optional Stopping Theorem,

$$[\,\eta > \tau\,]\mathbf{E}\,(N_\eta \,|\, \mathscr{F}_\tau) = [\,\eta > \tau\,] N_\tau$$

so the right-hand vanishes, i.e.

$$\mathbf{E}\left(\bar{N}_\eta \,|\, \mathscr{F}_\tau\right) = 0$$

which implies then that

$$\mathbf{E}\,\bar{N}_\eta = \mathbf{E}\,\mathbf{E}\left(\bar{N}_\eta \,|\, \mathscr{F}_\tau\right) = 0 = \mathbf{E}\,\bar{N}_0$$

Now Lemma 5.2. implies that $\bar{N}$ is martingale.

2. Let N and M be continuous local martingales. Show that

$$\langle\, M, N \,\rangle = \langle\, N, M \,\rangle$$

(Hint: use Theorem 6.10 and uniqueness)
Suggestion. Let $A = \langle\, M, N \,\rangle$. Since $MN - A = NM - A$ and $MN - A$ is continuous local martingale, we have by uniqueness that $A = \langle\, N, M \,\rangle$.

3. Let N_1, N_2 and M be continuous local martingales and $\alpha \in \mathbb{R}$. Show that

$$\langle\, M, N_1 + \alpha N_2 \,\rangle = \langle\, M, N_1 \,\rangle + \alpha\langle\, M, N_2 \,\rangle$$

(Hint: use Theorem 6.10 and uniqueness)
Suggestion. Let $A = \langle\, M, N_1 \,\rangle + \alpha\langle\, M, N_2 \,\rangle$. Since $M(N_1 + \alpha N_2) - A = MN_1 - \langle\, M, N_1 \,\rangle + \alpha M N_2 - \alpha\langle\, M, N_2 \,\rangle$ and the right-hand side is a continuous local martingale by Theorem 6.10, we have by uniqueness that $A = \langle\, M, N_1 + \alpha N_2 \,\rangle$.

Note: In the following two Problems, you may assume that it is known that

$$K_1 \cdot (K_2 \cdot H) = K_1 K_2 \cdot H$$

for locally bounded processes K_1, K_2 and $H \in \mathscr{A}$.

4. Let K and H be locally bounded processes (Definition 6.18) and let M be a continuous local martingale. Let $Y = H \cdot M$ be a continuous local martingale. Show that

$$K \cdot Y = KH \cdot M$$

(Hint: take N a continuous local martingale and use Theorem 6.20 twice to express $\langle K \cdot Y, N \rangle$ as an stochastic integral with respect to a process $\langle M, N \rangle$, and then use uniqueness in Theorem 6.20).

Suggestion. Let N be a continuous local martingale. By Theorem 6.20. we have that

$$\langle K \cdot Y, N \rangle \; = K \cdot \langle Y, N \rangle \; = K \cdot \langle H \cdot M, N \rangle \; = K \cdot (H \cdot \langle M, N \rangle).$$

Moreover, by Theorem 6.20. we also have that

$$\langle KH \;\; \cdot M, N \rangle = KH \langle M, N \rangle.$$

We see that the right-hand side coincides since

$$K \cdot (H \cdot \langle M, N \rangle) = KH \cdot \langle M, N \rangle.$$

Therefore, the left-hand side coincide, i.e.

$$\langle K \cdot Y, N \rangle \; = \langle KH \;\; \cdot M, N \rangle$$

which by the uniqueness in Theorem 6.20. implies that

$$K \cdot Y \;\; = KH \;\; \cdot M$$

as claimed.

5. Let K and H be locally bounded processes and M and N be continuous local martingales. Show that

$$\langle K \cdot M, H \cdot N \rangle = KH \cdot \langle M, N \rangle.$$

Suggestion. By previous Problem,

$$\langle K \cdot M, H \cdot N \rangle = K \cdot \langle M, H \cdot N \rangle.$$

By Problem 2. we know that the brackets are symmetric, and therefore,

$$\langle M, H \cdot N \rangle = \langle H \cdot N, M \rangle = H \cdot \langle N, M \rangle = H \cdot \langle M, N \rangle$$

again by the previous Problem. Combining these we see that

$$\langle K \cdot M, H \cdot N \rangle = K \cdot (H \cdot \langle M, N \rangle) = KH \cdot \langle M, N \rangle.$$

6. Let $X = \alpha K \cdot B$, where B is 1-dimensional Brownian motion and $K_t = B_t^2$. Verify that

$$\langle X, X \rangle_t = \alpha^2 \int_0^t B_s^4 \,\mathrm{d}s$$

(Hint: use Problem 5 and the fact that $t = \langle B, B \rangle_t$).
Suggestion. By Problem 5, we have that

$$\langle X, X \rangle_t = \langle \alpha K \cdot B, \alpha K \cdot B \rangle_t = \alpha^2 (K^2 \cdot \langle B, B \rangle)_t$$

Since $K^2 = B^4$ and $\langle B, B \rangle_t = t$, we have that

$$(K^2 \cdot \langle B, B \rangle)_t = \int_0^t B_s^4 \,\mathrm{d}s$$

which implies the claim.

7. Determine continuous local martingale M and locally finite variation process $A \in \mathscr{A}$, such that $A_0 = M_0 = 0$ and

$$B_t^4 = B_0^4 + M_t + A_t$$

where B is 1-dimensional Brownian motion. (Hint: $B_t^4 = f(B_t)$ when $f(x) = x^4$ and Itō's formula.)
Suggestion. Let's use Itō's formula for $f(x) = x^4$. Since $f'(x) = 4x^3$ and $f''(x) = 12x^2$, we have by the Itō's formula that

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) \,\mathrm{d}B_s + \tfrac{1}{2} \int_0^t f''(B_s) \,\mathrm{d} \langle B, B \rangle_s$$

Now, the first term is a continuous local martingale

$$M_t = \int_0^t f'(B_s) \,\mathrm{d}B_s = 4 \int_0^t B_s^3 \,\mathrm{d}B_s$$

and the latter term is of locally finite variation

$$A_t = \tfrac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}\langle B, B\rangle_s = 6\int_0^t B_s^2\,\mathrm{d}s$$

8. Use Itō's formula to find a polynomial function $f(x, y)$ such that

$$B_t^6 - f(B_t, t)$$

is a continuous local martingale. (Hint: Let $f_0(x, y) = x^6 + c_1x^4y + c_2x^2y^2 + c_3y^3$. With Itō obtain a equations for the coefficients c_1, c_2, c_3)
Suggestion. Let's compute the partial derivatives of f_0 first. To make it clear what the partial derivatives are we will denote them by ∂_1, ∂_2, etc. for differentiation with respect to first, second, etc. variables. We have

$$\begin{aligned}\partial_1 f_0(x, y) &= 6x^5 + 4c_1x^3y + 2c_2xy^2\\ \partial_2 f_0(x, y) &= c_1x^4 + 2c_2x^2y + 3c_3y^2\\ \partial_1^2 f_0(x, y) &= 30x^4 + 12c_1x^2y + 2c_2y^2\end{aligned}$$

We don't need to compute $\partial_1\partial_2 f_0$ nor $\partial_2^2 f_0$, since $t \mapsto t$ is of locally finite variation. Using Itō with $f_0(B_t)$ we therefore have,

$$f_0(B_t, t) = f_0(B_0, 0) + \int_0^t \partial_1 f_0(B_s, s)\,\mathrm{d}B_s + \int_0^t \partial_2 f_0(B_s, s)\,\mathrm{d}s + \tfrac{1}{2}\int_0^t \partial_1^2 f_0(B_s, s)\,\mathrm{d}s.$$

Therefore, the right-hand side is a local martingale, if

$$\partial_2 f_0 + \tfrac{1}{2}\partial_1^2 f_0 = 0$$

or

$$\begin{cases} c_1 + \frac{1}{2}\times 30 = 0\\ 2c_2 + \frac{1}{2}\times 12c_1 = 0\\ 3c_3 + \frac{1}{2}\times 2c_2 = 0\end{cases}$$

This has a solution

$$\begin{cases} c_1 = -15\\ c_2 = 45\\ c_3 = -15\end{cases}$$

This means that

$$f_0(B_t, t) = B_t^6 - 15B_t^4 t + 45B_t^2 t^2 - 15t^3$$

is a local martingale, so by choosing

$$f(x, y) = 15x^4 y - 45x^2 y^2 + 15y^3$$

the mission is accomplished.

Note. We say that a process X is in class (DL), if $\{ X_\tau : \tau$ is a *bounded* stopping time $\}$ is uniformly integrable.

9. Let M be a continuous local martingale. Show that if M is in class (DL), then it is a martingale. (Hint: Let (τ_n) is the sequence as in the Definition 5.10. and let τ be a bounded stopping time. Show that $M_{\tau_n \wedge \tau} \to M_\tau$ almost surely, that $\mathbf{E}_x M_{\tau_n \wedge \tau} = \mathbf{E}_x M_0$ and that $\{M_{\tau_n \wedge \tau}\}_n$ is uniformly integrable. Then have a look at Lemma 5.2. in lecture notes and the Problem 4 in the Excercise sheet 3.)
Suggestion. As in the hint, let (τ_n) is the sequence as in the Definition 5.10. and let τ be a bounded stopping time. Now since $\tau_n \uparrow \infty$ almost surely, we obtain that

$$\tau \wedge \tau_n \uparrow \tau$$

almost surely and hence by continuity (which was missing from the assumptions first) we have that

$$M_{\tau_n \wedge \tau} \to M_\tau$$

almost surely. Now if we can show that $\mathbf{E}_x M_\tau = \mathbf{E}_x M_0$ we have by the Lemma 5.2. that M is a martingale. Since M^{τ_n} is a uniformly integrable martingale for every n, we can use the Optional Stopping theorem and we deduce that

$$\mathbf{E}_x M_\tau^{\tau_n} = \mathbf{E}_x M_0^{\tau_n} = \mathbf{E}_x M_0 .$$

Since $M_\tau^{\tau_n} = M_{\tau_n \wedge \tau}$, we have that

$$\mathbf{E}_x M_{\tau_n \wedge \tau} = \mathbf{E}_x M_0$$

for every n and so, if we can can change the order of limit $n \to \infty$ and the expectation, we would obtain

$$\mathbf{E}_x M_0 = \lim_{n\to\infty} \mathbf{E}_x M_{\tau_n \wedge \tau} = \mathbf{E}_x \lim_{n\to\infty} M_{\tau_n \wedge \tau} = \mathbf{E}_x M_\tau$$

which would then imply the claim by Lemma 5.2. According to Problem 4 in the Excercise sheet 3, we can change the order of the limit and the expectation, if $\{M_{\tau_n \wedge \tau}\}_n$ is uniformly integrable. However, since $\tau_n \wedge \tau$ is a bounded stopping time, the (DL) assumption shows that $\{M_{\tau_n \wedge \tau}\}_n$ is uniformly integrable which finishes the proof.

10. Let $X_t = |B_t|^{-1}$ where B_t is *three-dimensional* Brownian motion that starts from $x \neq 0$. In lectures we showed that X_t is a local martingale. Show that it is not in class (DL). (Hint: show directly that it is *not a martingale* by considering the mean $\mathbf{E}\, X_t$ of X_t.)
Suggestion. It is enough to show that $m(t) := \mathbf{E}_x\, X_t \ = \mathbf{E}_x\, |B_t|^{-1}$ is not constant function, since if X would be a martingale, then m would be a constant. In order to accomplish this, we will show the following properties:

- $m(0) = |x|^{-1}$
- $\lim_{t\to\infty} m(t) = 0.$

This means that $m(t) < \frac{1}{2}|x|^{-1}$ for large t, i.e. m is not a constant. Hence X cannot be a martingale and but since it is a continuous local martingale, it cannot be of class (DL) by Problem 9.

The property that $m(0) = |x|^{-1}$ is immediate, since

$$m(0) = \mathbf{E}_x\, |B_0|^{-1} \ = |x|^{-1}$$

Just to make this example more interesting, we first show that $\mathbf{E}_x\, X_t^2 \ < \infty$. This means that local martingale which is even twice integrable does not have to be martingale.

The finiteness of the second moment follows since

$$\mathbf{E}_x\, X_t^2 \ = \mathbf{E}_0\, |B_t + x|^{-2} \ \leq M^{-2} + \mathbf{E}_0\, (|B_t + x|^{-2}[\, |B_t + x| \leq M\,])$$

Since we know the probability transition density of Brownian motion, we can express the latter expectation as

$$(2\pi t)^{-3/2} \int_{|y+x|\leq M} |x+y|^{-2} e^{-|y|^2/2t}\, \mathrm{d}y.$$

We will just estimate the exponential function with a constant, which then yields the following estimate

$$(2\pi t)^{-3/2} \int_{|y+x|\leq M} |x+y|^{-2} e^{-|y|^2/2t}\, \mathrm{d}y \lesssim t^{-3/2} \int_{|y|\leq M} |y|^{-2}\, \mathrm{d}y = 4\pi t^{-3/2} \int_0^M r^{-2} r^2\, \mathrm{d}r$$

where we then used integration in the polar coordinates (note, that is the reason for the appearence of Jacobian determinant r^2 in the last integral). So, we have deduced that for every $M > 0$ we have

$$\mathbf{E}_x X_t^2 = \mathbf{E}_0 |B_t + x|^{-2} \le M^{-2} + cMt^{-3/2}$$

and so the second moment is uniformly bounded for $t \ge t_0 > 0$.

Now the Cauchy–Schwarz inequality gives that

$$\mathbf{E}_x X_t \le \sqrt{\mathbf{E}_x X_t^2} \le \sqrt{M^{-2} + cMt^{-3/2}}$$

which shows that

$$\limsup_{t\to\infty} \mathbf{E}_x X_t \le M^{-1}$$

for every $M > 0$. Letting $M \to \infty$ implies that $m(t) \to 0$ as $t \to \infty$.

11. We know that for every $x \neq 0$ in the plane (i.e. in $\mathbb{R}^2$) that

$$\mathbf{P}_x (\tau_0 = \infty) = 1$$

where x is the starting point of *two-dimensional* Brownian motion. Show that this holds also for $x = 0$. (Hint: let $\nu = \tau_r$ be the first hitting time to the sphere of radius r and use strong Markov property at ν. Use this to deduce the claim.)
Suggestion. Let ν be as in the hint. If we assume that $\tau_0 = \infty$, then $\nu < \tau_0$, since we know that $\nu < \infty$. Therefore, we may use the strong Markov property and we obtain that

$$\mathbf{P}_x (\tau_0 = \infty) = \mathbf{E}_x [\tau_0 = \infty, \nu < \tau_0] = \mathbf{E}_x [\nu < \tau_0] \mathbf{P}_{B_\nu} (\tau_0 = \infty) = \mathbf{E}_x \mathbf{P}_{B_\nu} (\tau_0 = \infty)$$

Since we know that $B_\nu \neq 0$ almost surely, we know that $\mathbf{P}_{B_\nu} (\tau_0 = \infty) = 1$ almost surely. Therefore,

$$\mathbf{P}_x (\tau_0 = \infty) = \mathbf{E}_x 1 = 1$$

and the claim follows.

12. Suppose there exists a 1-dimensional continuous semimartingale X such that

$$\mathrm{d}X_t = a(X_t)\, \mathrm{d}B_t + b(X_t)\, \mathrm{d}t$$

where a and b are bounded and $C^2(\mathbb{R},\mathbb{R})$-functions. Let $Z = f(X_t)$ for $f \in C^2(\mathbb{R},\mathbb{R})$. Use Itō's formula to find a continuous local martingale M and process $A \in \mathscr{A}$ such that $A_0 = M_0 = 0$ and

$$Z_t = Z_0 + M_t + A_t$$

Furthermore, compute $\langle M, M \rangle$ (Hint. Use the formula in Problem 5 for the computation).

Suggestion. By Itō's formula

$$Z_t = Z_0 + \int_0^t f'(X_t)\,\mathrm{d}X_t + \tfrac{1}{2}\int_0^t f''(X_t)\,\mathrm{d}\langle X, X \rangle_t$$

Since $\mathrm{d}X_t = a(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t$, we see that if we denote $K_1(t) = a(X_t)$ and $K_2(t) = f'(X_t)$, the local martingale part M is

$$M_t = (K_1 \cdot (K_2 \cdot B))_t = (K_1K_2 \cdot B)_t = \int_0^t K_1(s)K_2(s)\,\mathrm{d}B_s = \int_0^t a(X_s)f'(X_s)\,\mathrm{d}B_s.$$

This also implies with Problem 5 that

$$\langle M, M \rangle_t = K_1^2K_2^2 \cdot H_t = \int_0^t a(X_s)^2 f'(X_s)^2\,\mathrm{d}s.$$

The locally finite variation process A is by the Itō formula the remainding terms, i.e.

$$A_t = \int_0^t b(X_s)f'(X_s)\,\mathrm{d}s + \tfrac{1}{2}\int_0^t f''(X_s)\,\mathrm{d}\langle X, X \rangle_s$$

where the first term is deduced by associativity property $K_3 \cdot (K_2 \cdot H) = K_3K_2 \cdot H$ where $K_3(t) = b(X_t)$ and $H_t = t$. Since bracket $\langle X, H \rangle = 0$ for every semimartingale X and locally finite variation process H, we have $\langle X, X \rangle = \langle K_1 \cdot B, K_1 \cdot B \rangle$. Therefore, by Problem 5

$$\langle X, X \rangle = K_1^2 \cdot H$$

and so

$$A_t = \int_0^t b(X_s)f'(X_s)\,\mathrm{d}s + \tfrac{1}{2}\int_0^t f''(X_s)a(X_s)^2\,\mathrm{d}s$$