Department of Mathematics and Statistics
Stochastic processes on domains
Suggestions to excercise problem sheet 3

1. Suppose τ_1 and τ_2 are $(\mathscr{F}_{t^+})$-stopping times and $\tau_1 \le \tau_2$. Show that $\mathscr{F}_{\tau_1^+} \subset \mathscr{F}_{\tau_2^+}$.
Suggestion. There was an unfortunate misprint and the essential assumption $\tau_1 \le \tau_2$ was missing.

With this assumption, the proof might go as follows. Suppose $A \in \mathscr{F}_{\tau_1^+}$. Then we know that for every $t > 0$ the event $\{\tau_1 \le t, A\} \in \mathscr{F}_{t^+}$.

Since $\tau_1 \le \tau_2$, we have that $\{\tau_1 \le t, \tau_2 \le t\} \supset \{\tau_2 \le t\}$. The other inclusion holds always, and so the events $\{\tau_1 \le t, \tau_2 \le t\} = \{\tau_2 \le t\}$. Therefore,

$$\{A, \tau_2 \le t\} = \{A, \tau_1 \le t\} \cap \{\tau_2 \le t\}.$$

Since τ_2 is $(\mathscr{F}_{t^+})$-stopping time, we have that $\{\tau_2 \le t\} \in \mathscr{F}_{t^+}$ and so

$$\{A, \tau_2 \le t\} \in \mathscr{F}_{t^+}$$

for every t. This, however, means that $A \in \mathscr{F}_{\tau_2^+}$ and the claim follows.

2. Let X be a positive and integrable random variable. Show that the family $\{\, Y : \ |Y| \ \le X \,\}$ is uniformly integrable. (Hint: estimate $|Y|[\,|Y| > m\,]$ from above by random variable depending only on X and m).
Suggestion. Let's denote the set $\{\, Y : \ |Y| \ \le X \,\}$ with $\mathscr{C}$ and let's follow the hint by estimating $|Y|[\,|Y| > m\,]$ from above. Let $Y \in \mathscr{C}$. Since $|Y| \ \le X$, we have

$$\mathbf{E}\,|Y|[\,|Y| > m\,] \le \mathbf{E}\,X[\,|Y| > m\,]\,.$$

Since $|Y| \le X$, then $[\,|Y| > m\,] \ \le [\,X > m\,]$. This implies that

$$\mathbf{E}\,|Y|[\,|Y| > m\,] \le \mathbf{E}\,X[\,X > m\,]\,.$$

Since this holds for every $Y \in \mathscr{C}$ we obtain

$$\sup_{Y \in \mathscr{C}} \mathbf{E}\,|Y|[\,|Y| > m\,] \le \mathbf{E}\,X[\,X > m\,]$$

and therefore,

$$\limsup_{m\to\infty}\sup_{Y\in\mathscr{C}}\mathbf{E}\,|Y|[\,|Y|>m\,]\;\le\limsup_{m\to\infty}\mathbf{E}\,X[\,X>m\,]\,.$$

Since $X[\,X>m\,]\le X$ and $X[\,X>m\,]\to 0$ almost surely, we have by the Lebesgue Dominated Convergence theorem that

$$\limsup_{m\to\infty}\mathbf{E}\,X[\,X>m\,]\;=0$$

and therefore the claim follows.

3. Assume that $\mathbf{E}\,X_n^2\;\le M$ for every n. Show that $\{\;X_n:\;\;n\in\mathbb{N}\;\}$ is uniformly integrable. (Hint: $|X_n|m\le X_n^2$ when $|X_n|\ge m$.)
Suggestion. Let's denote $\mathscr{C}\;=\{\;X_n:\;\;\mathbf{E}\,X_n^2\;\le M\;\}$. Now we have by the hint above that

$$\mathbf{E}\,|X_n|[\,|X_n|>m\,]\;=\frac{1}{m}\mathbf{E}\,|X_n|m[\,|X_n|>m\,]\;\le\frac{1}{m}\mathbf{E}\,X_n^2[\,|X_n|>m\,]$$

Since $\mathbf{E}\,X_n^2[\,|X_n|>m\,]\;\le\mathbf{E}\,X_n^2\;\le M$, we have obtained an upper bound

$$\sup_{X_n\in\mathscr{C}}\mathbf{E}\,|X_n|[\,|X_n|>m\,]\;\le\frac{M}{m}.$$

This implies the claim, since $\frac{M}{m}\to 0$ when $m\to\infty$.

Note that we didn't actually need the countability of the set $\mathscr{C}$ so we could have used $\mathscr{C}=\{\;X:\;\;\mathbf{E}\,X^{\,2}\le M\;\}$ equally well. Furthermore, the second power is not essential, since if $\mathscr{C}_p=\{\;X:\;\;\mathbf{E}\,|\,X|^p\le M\;\}$, then the same proof idea gives that

$$\mathbf{E}\,|X|[\,|X|>m\,]\;=m^{1-p}\mathbf{E}\,|X|m^{p-1}[\,|X|>m\,]\;\le Mm^{1-p}.$$

This means that $\mathscr{C}_p$ is uniformly integrable for every $p>1$ but the technique fails when $p\le 1$. However, just above $p=1$ we are fine since $\mathscr{C}_\phi=\{\;X:\;\;\mathbf{E}\,X\,\phi(X)\le M\;\}$ is uniformly integrable, as long as $\phi\uparrow\infty$, so for instance rather slowly increasing function $\phi(x)\;=[\,x>e^{e^e}\,]\log\log\log x$ works equally well. For instance $\phi(10^{729})=3\log\log 10^9=3*3\log 10^3=27\log 10\le 65$.

4. Assume $X_n\to X$ in almost surely and $\{\;X_n:\;\;n\in\mathbb{N}\;\}$ is uniformly integrable. Show that $X_n\to X$ in L^1-sense. (Hint: let $\phi_n(x)=x[\,|x|\le n\,]+n[\,x>n\,]-n[\,x<$

$-n\,]$ and write $X_k - X = (X_k - \phi_n(X_k)) - (X - \phi_n(X)) + (\phi_n(X_k) - \phi_n(X)) = I_1 + I_2 + I_3$ and estimate term I_j each separetely).

Suggestion. Let's try to follow the hint. This means that

$$\mathbf{E}\,|X_k - X| \,\leq \mathbf{E}\,|I_1(n,k)| \,+ \mathbf{E}\,|I_2(n)| \,+ \mathbf{E}\,|I_3(n,k)|\,.$$

The third term is something that vanishes for *fixed* n, since $|\phi_n(X_k) - \phi_n(X)| \leq 2n$ is bounded and $|\phi_n(X_k) - \phi_n(X)| \to 0$ almost surely, as $k \to \infty$. Therefore,

$$\lim_{k\to\infty} \mathbf{E}\,|I_3(n,k)| \,= 0$$

for every fixed n.

The first term is handled with uniform integrability, since

$$|I_1(n,k)| \leq |X_k| \,[\,|X_k| > n\,]$$

and so for every large enough n, say $n \geq n_1$, we have

$$\sup_k \mathbf{E}\,|I_1(n,k)| \,\leq \varepsilon.$$

The second term can again be estimated as

$$|I_2(n)| \leq |X| \,[\,|X| > n\,]$$

and if we can verify that $\mathbf{E}\,|X| \,< \infty$, then for every n large enough, say $n \geq n_2$, we would have

$$\mathbf{E}\,|I_2(n,k)| \,\leq \varepsilon.$$

So once we have verified that X is integrable, then by choosing $n = n_1 \vee n_2$, we have for every k that

$$\mathbf{E}\,|X_k - X| \,\leq 2\varepsilon + \mathbf{E}\,I_3(n,k)$$

or

$$\limsup_{k\to\infty} \mathbf{E}\,|X_k - X| \,\leq 2\varepsilon.$$

So the claim follows if we can verify that X is integrable. For this we use Fatou's lemma, which says that

$$\mathbf{E}\,|X| \,\leq \liminf_{n\to\infty} \mathbf{E}\,|X_n| \,= \sup_{n\geq 1} \inf_{k\geq n} \mathbf{E}\,|X_n| \,\leq \sup_{n\geq 1} \mathbf{E}\,|X_n|$$

The term on the right is bounded from above by the uniform integrability, since suppose

$$\sup_n \mathbf{E}\,|X_n|[\,|X_n|\; > M\,] \le 1$$

for some M. Then

$$\mathbf{E}\,|X_n| \;= \mathbf{E}\,|X_n|[\,|X_n|\; \le M\,] + \mathbf{E}\,|X_n|[\,|X_n|\; > M\,] \le M + 1$$

and therefore,

$$\mathbf{E}\,|X| \;\le \sup_n \mathbf{E}\,|X_n| \;\le M + 1.$$

So X is integrable, which finishes the proof.

5. Let $X_t = B_{t\wedge s}$ be a stopped 1-dimensional Brownian motion at time instance $s \in (0,\infty)$ (i.e. at constant stopping time). Show that $X_t = \mathbf{E}\left(X_\infty \,|\, \mathscr{F}_t\right)$ for every $t \in (0,\infty)$ and deduce that X is uniformly integrable martingale. (Corrected Hint: X_t and $-X_t$ are supermartingales if X is a martingale, see Lemma 5.2. and Theorem 5.3.)

Suggestion. Let's verify that X is a martingale that satisfies $X_t = \mathbf{E}\left(X_\infty \,|\, \mathscr{F}_t\right)$. This we have done already in Excercise sheet 2 Problem 8. provided we show that X_∞ is integrable. But since

$$\mathbf{E}\,|X_\infty| \;= \mathbf{E}\,|B_s| \;< \infty$$

the martingale property follows form the Excercise sheet 2 Problem 8. Another way would have been the direct computation or using Lemma 5.2.

Next we want to show X is uniformly integrable which shows that a Brownian motion which is stopped at a constant time is uniformly integrable and therefore shows that Brownian motion is also a local martingale. The same proof shows that every martingale is a local martingale.

This follows from the formulation of Theorem 5.3. for martingales which are both supermartingales and submartingales. Than one of the claims would be that X is uniformly integrable martingale if and only if $X_t = \mathbf{E}\left(X_\infty \,|\, \mathscr{F}_t\right)$ for some integrable random variable X_∞. But let's verify this claim in detail, since it is not in the lecture notes.

Since X is a martingale, X is also a supermartingale and therefore,

$$X_t^- = \mathbf{E}\left(X_\infty \,|\, \mathscr{F}_t\right)^- \le \mathbf{E}\left(X_\infty^- \,|\, \mathscr{F}_t\right)$$

by Jensen's inequality. Furthermore, $Y = -X$ is a martingale and hence a super-martingale with

$$Y_t^- \leq \mathbf{E}\left(Y_\infty^- \,|\, \mathscr{F}_t\right).$$

Now The 5.3. says that both $\mathscr{C}_- := \{ X_t^- : t \geq 0 \}$ and $\mathscr{C}_+ := \{ Y_t^- : t \geq 0 \}$ are uniformly integrable. However, since $(-x)^- = [-x < 0](- - x) = [x > 0]x = x^+$, we therefore have $Y_t^- = X_t^+$ and so $\mathscr{C}_+ := \{ X_t^+ : t \geq 0 \}$. This, however, implies that $\mathscr{C} := \{ X_t : t \geq 0 \}$ is uniformly integrable, since

$$\mathbf{E}\,|X_t|\,[\,|X_t| > K\,] = \mathbf{E}\,X_t^+[\,X_t^+ > K\,] + \mathbf{E}\,X_t^-\,[\,X_t^- > K\,] \leq \varepsilon$$

when K is large enough.

6. Define

$$p_0(t,x,y) = \frac{1}{\sqrt{2\pi t}}(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}})$$

for every $t > 0$, and $x, y \in \mathbb{R}$ and $x, y \geq 0$. Verify that

$$\partial_t p_0(t,x,y) = \tfrac{1}{2}\partial_y^2 p_0(t,x,y) \quad \text{and} \quad p_0(t,0,y) = 0$$

for every $t > 0$ and $x, y > 0$. This is the transition probability density function of 1-dimensional killed BM which is killed at zero.

Suggestion. Let's rewrite $p_0(t,x,y) = q(t, y-x) - q(t, y+x)$ where $q(t,y) = (2\pi t)^{-1/2}\exp(-1/(2t)y^2)$. Now we will first test the initial value condition $p_0(t,0,y) = 0$ first. Since

$$p_0(t,0,y) = q(t, y-0) - q(t, y+0) = 0$$

this follows immediately. Now we need to do the differentiation. Since $q(t,y)$ is positive, we can compute its logarithm $r = \log q$ and

$$r(t,y) = -\tfrac{1}{2}\log(2\pi) - \tfrac{1}{2}\log t - \frac{y^2}{2t}$$

Now $\partial_t q = q\partial_t r$, so

$$\partial_t q(t,y) = \tfrac{1}{2}q(t,y)(-t^{-1} + y^2 t^{-2})$$

In the same way

$$\partial_y q(t,y) = \tfrac{1}{2}q(t,y)(2y)t^{-1} = q(t,y)yt^{-1}$$

The second derivative is then

$$\partial_y^2 q(t,y) = yt^{-1}\partial_y q(t,y) + q(t,y)t^{-1} = \big(y^2 t^{-2} + t^{-1}\big)q(t,y) = 2\partial_t q(t,y)$$

or $\partial_t q = \frac{1}{2}\partial_y^2 q$. Using this we get the original claim, but to make it easier to follow, let's denote $\partial_t q = \partial_1 q$ and $\partial_y^2 q = \partial_2^2 q$, which then denote the derivative with respect to first and second variable.

Therefore,

$$\begin{aligned}\partial_t p_0(t,x,y) &= \partial_t q(t,y-x) - \partial_t q(t,y+x) = (\partial_1 q)(t,y-x) - (\partial_1 q)(t,y+x) \\ &= \tfrac{1}{2}(\partial_2^2 q)(t,y-x) - \tfrac{1}{2}(\partial_2^2 q)(t,y+x)\end{aligned}$$

and on the other hand

$$\partial_y^2 p_0(t,x,y) = \partial_y(\partial_2 q)(t,y-x) - \partial_y(\partial_2 q)(t,y+x) = (\partial_2^2 q)(t,y-x) - (\partial_2^2 q)(t,y+x)$$

so comparing the right-hand side we get the claim.

Note that if we would have used $p_0(t,x,y) = q(t,x-y) - q(t,x+y)$ formulation, then $\partial_y p_0(t,x,y) = -(\partial_2 q)(t,x-y) - (\partial_2 q)(x+y)$ but the second derivative would have changed the sign back, since $-\partial_y(\partial_2 q)(t,x-y) = (\partial_2 q)(t,x-y)$.

7. Let τ be a simple stopping time, say $\tau \in \{t_1,\dots,t_n\}$. Show that X_τ is $\mathscr{F}_\tau$-measurable and $\mathscr{F}_\tau = \mathscr{F}_{\tau+}$.
Suggestion. Let $U = \{X_\tau \in A\}$.

$$[\,X_\tau \in A, \tau \le t\,] = \sum_{t_j \le t} [\,X_{t_j} \in A, \tau = t_j\,]$$

we see that $U \in \mathscr{F}_{\tau+}$. But since A is arbitrary, X_τ is $\mathscr{F}_{\tau+}$-measurable. The second claim is valid for every τ, since since if $A \in \mathscr{F}_\tau$, then

$$\{A, \tau \le t\} \in \mathscr{F}_t \subset \mathscr{F}_{t+}$$

and if $A \in \mathscr{F}_{\tau+}$, then

$$\{A, \tau \le t\} \in \mathscr{F}_{t+} \subset \mathscr{F}_t$$

by right-continuity.

8. Let $s \le t < u$ and τ a stopping time. Show that

$$\mathbf{E}\left([\tau > t](X_u - X_t) \,|\, \mathscr{F}_s\right) \le 0$$

(Hint: You can use the fact that $\mathbf{E}\left(Z \,|\, \mathscr{H}\right) = \mathbf{E}\left(\mathbf{E}\left(Z \,|\, \mathscr{G}\right) \,|\, \mathscr{H}\right)$ for every σ-algebras $\mathscr{H} \subset \mathscr{G}$ and then you can apply the supermartingale property).

Suggestion. Since $s \le t$, we have that $\mathscr{F}_s \subset \mathscr{F}_t$ and so

$$\mathbf{E}\left([\,\tau > t\,](X_u - X_t)\,|\,\mathscr{F}_s\right) = \mathbf{E}\left(\mathbf{E}\left([\,\tau > t\,](X_u - X_t)\,|\,\mathscr{F}_t\right)\ |\,\mathscr{F}_s\right).$$

Since $[\,\tau > t\,]$ is $\mathscr{F}_{t+} = \mathscr{F}_t$-measurable, we have

$$\mathbf{E}\left([\,\tau > t\,](X_u - X_t)\,|\,\mathscr{F}_t\right) = [\,\tau > t\,]\mathbf{E}\left(X_u - X_t\,|\,\mathscr{F}_t\right) \le 0$$

by the supermartingale property. Therefore,

$$\mathbf{E}\left([\,\tau > t\,](X_u - X_t)\,|\,\mathscr{F}_s\right) = \mathbf{E}\left(\mathbf{E}\left([\,\tau > t\,](X_u - X_t)\,|\,\mathscr{F}_t\right)\ |\,\mathscr{F}_s\right) \le 0$$

as well.

9. Let τ and be a simple stopping time, say $\tau \in \{t_1, \dots, t_n\}$. Let us assume that $s = t_0 \le t_1 < t_2 < \cdots < t_n$. Show that and

$$\mathbf{E}\left(X_\tau\,|\,\mathscr{F}_s\right) \le X_s.$$

(Hint: show

$$X_\tau - X_s = \sum_{k=0}^{n-1}[\,\tau > t_k\,](X_{t_{k+1}} - X_{t_k})$$

and use this identity with Problem 8.)

Suggestion. Suppose we have shown the identity of the hint. Then

$$\mathbf{E}\left(X_\tau - X_s\,|\,\mathscr{F}_s\right) = \sum_{k=0}^{n-1}\mathbf{E}\left([\,\tau > t_k\,](X_{t_{k+1}} - X_{t_k})\,|\,\mathscr{F}_s\right) \le 0$$

since using $s = s$, $t = t_k$ and $u = t_{k+1}$ we have exactly the case of Problem 8. But hen the claim follows, since

$$\mathbf{E}\left(X_\tau\,|\,\mathscr{F}_s\right) = X_s + \mathbf{E}\left(X_\tau - X_s\,|\,\mathscr{F}_s\right) \le X_s$$

where we used that X_s is $\mathscr{F}_s$-measurable. So we only need to show the identity. Since τ is simple stopping time, we can write

$$X_\tau - X_s = \sum_{k=0}^{n}[\,\tau = t_k\,](X_{t_k} - X_{t_0}) = \sum_{k=1}^{n}[\,\tau = t_k\,]\sum_{j=0}^{k-1}(X_{t_{j+1}} - X_{t_j})$$

where we wrote the difference as a telescoping sum. Let's change the order of summations and we get

$$X_\tau - X_s = \sum_{j=0}^{n-1}\sum_{k=j+1}^{n}[\,\tau = t_k\,](X_{t_{j+1}} - X_{t_j}).$$

Since

$$\sum_{k=j+1}^{n} [\,\tau = t_k\,] = [\,\tau \in \{t_{j+1}, \dots, t_n\}\,] = [\,\tau \geq t_{j+1}\,] = [\,\tau > t_j\,]$$

the claim follows.

10. Let $X_t = e^{sB_t - s^2t/2}$. Show that it is a martingale with respect to the history of Brownian motion. Hint: you are on right track if you have arrived to

$$\mathbf{E}\,(X_u \,|\, \mathscr{F}_t) \;= e^{sB_t} e^{-s^2u/2} \mathbf{E}\, e^{sB_{u-t}}$$

Suggestion. First we verify that X is adapted, but this is evident, since $X_t = f(B_t)$ for $f(x) = e^{sx}e^{-s^2t/2}$ which is continuous function.

Next we want to make sure that X_t is integrable. Since $X_t \geq 0$, we have

$$\begin{aligned}\mathbf{E}_0\, X_t &= c_t \int_{\mathbb{R}} e^{sx} e^{-s^2t/2} e^{-x^2/2t}\;\mathrm{d}x = c_t \int_{\mathbb{R}} e^{-\frac{1}{2t}(x^2 - 2stx + s^2t^2)}\,\mathrm{d}x \\ &= c_t \int_{\mathbb{R}} e^{-\frac{x^2}{2t}}\,\mathrm{d}x = 1\end{aligned}$$

which also shows that the mean is constant, show we might have a martingale. But to verify that let's compute the conditional expectation

$$\mathbf{E}_0(X_u \,|\, \mathscr{F}_t) \;= e^{sB_t - s^2u/2}\mathbf{E}_0\Big(e^{s(B_u - B_t)} \,|\, \mathscr{F}_t\Big) \;= e^{sB_t} e^{-s^2u/2}\mathbf{E}\, e^{sB_{u-t}}$$

since $B_u - B_t$ is independent from $\mathscr{F}_t$ and $B_u - B_t \sim B_{u-t}$. Since $e^{sB_t} = X_t e^{s^2t/2}$, we have

$$\mathbf{E}_0(X_u \,|\, \mathscr{F}_t) \;= X_t e^{-s^2(u-t)/2}\mathbf{E}_0\, e^{sB_{u-t}} \;= X_t \mathbf{E}_0\, X_{u-t} \;= X_t$$

since the mean is 1.

11. Let $\tau_a = \inf\{\; t > 0 : \;\; B_t = a \;\}$ be the first hitting time of *1-dimensional* Brownian motion to the point a. Let $a < x < b$ and $X_t = B_t^{\tau_a \wedge \tau_b}$. Show that X_t is a bounded martingale for every $x \in (a, b)$ where x is the starting point of X.
Suggestion. Let's verify first that X is bounded, i.e. we want to show that

$$\sup_t |X_t| \;\leq M$$

almost surely. When $t < \tau$, we know that B_t has to be between a and b, since B_0 is. Therefore, when $X_0 = x$ we have

$$\sup_{t<\tau} |X_t| \;\leq M$$

almost surely where $M = |a| \vee |b|$. When $t \geq \tau$, then either $X_t = a$ or $X_t = b$, and so

$$\sup_t |X_t| \leq M$$

almost surely.

Furthermore, the Lemma 5.9. says that $X^\nu = B^{\tau_a \wedge \tau_b \wedge \nu}$ is a martingale for every bounded stopping time ν, since B is right-continuous martingale and $\tau_a \wedge \tau_b \wedge \nu$ is a bounded stopping time. Since X^ν is a bounded martingale, we have by Optional Stopping Theorem that

$$\mathbf{E}_x X_\nu = \mathbf{E}_x X_\nu^\nu = \mathbf{E}_x X_0$$

and hence Lemma 5.2. says that X is martingale itself.

12. Assume the same as in 11. Show that

$$a\mathbf{P}_x(\tau_a < \tau_b) + b\mathbf{P}_x(\tau_b < \tau_a) = x$$

and show that

$$\mathbf{P}_x(\tau_a < \tau_b) = \frac{b-x}{b-a} = 1 - \mathbf{P}_x(\tau_a > \tau_b).$$

(Hint: Optional Stopping Theorem for a bounded martingale. For the latter explain first why $\mathbf{P}_x(\tau_a = \tau_b) = 0$ and then you have two equations for the probabilities).

Suggestion. We know that $X = B^{\tau_a \wedge \tau_b}$ is a bounded martingale for every starting point $x \in (a, b)$ by Problem 11. Therefore, we can use the Optional Stopping Theorem and we get that

$$\mathbf{E}_x X_{\tau_a \wedge \tau_b} = \mathbf{E}_x X_0 = x.$$

This means that

$$x = \mathbf{E}_x B_{\tau_a \wedge \tau_b} = a\mathbf{P}_x(\tau_a < \tau_b) + b\mathbf{P}_x(\tau_b < \tau_a) + \infty \mathbf{P}_x(\tau_a = \tau_b = \infty)$$

since $\tau_a = \tau_b < \infty$ cannot happen, since Brownian motion cannot be at the two places at the same time. The third option must vanish, since left-hand side is finite. This verifies the first claim.

Also the vanishing of the third term implies $\mathbf{P}_x(\tau_a = \tau_b) = 0$ and so

$$\mathbf{P}_x(\tau_a < \tau_b) + \mathbf{P}_x(\tau_b < \tau_a) = 1.$$

Now we have two equations

$$\begin{cases} pa + qb = x \\ p + q = 1 \end{cases}$$

where $p = \mathbf{P}_x\left(\tau_a < \tau_b\right)$ and $q = \mathbf{P}_x\left(\tau_b < \tau_a\right)$ which has a unique solution

$$\begin{cases} p = \dfrac{b-x}{b-a} \\ q = \dfrac{x-a}{b-a} \end{cases}$$