Department of Mathematics and Statistics
Stochastic processes on domains
Suggestions to excercise problem sheet 2

Note. In the Problems 1-10 the j,k and n are always integers.

1. Let $(P_{t,x})$ be as in Lemma 3.14 in Lecture notes (page 23). Let

$$\mu^x_{(t_1,\dots,t_n)}(A_1 \times \dots \times A_n) = \int_{A_1} P_{t_1,x}(\,\mathrm{d}x_1) \dots \int_{A_n} P_{t_n-t_{n-1},x_{n-1}}(\,\mathrm{d}x_n)$$

for every $x \in S$, for every $n \in \mathbb{N}_+$, for every $t_1 < \dots < t_n$ and for every $A_1, \dots, A_n \in \mathscr{S}$. Let $(A_n) \subset \mathscr{S}$ be a sequence of sets in $\mathscr{S}$. Define $B_{k,j}, C_{k,j} \in \mathscr{S}$ as $[\,B_{k,j}\,] = [\,A_k\,][\,k \neq j\,] + [\,S\,][\,k = j\,]$. and $[\,C_{k,j}\,] = [\,A_k\,][\,k < j\,] + [\,A_{k-1}\,][\,k > j\,]$. Let $\pi_{j,n}(t_1, \dots, t_n) = (s_1, \dots, s_{n-1})$ where $s_k = t_k[\,k < j\,] + t_{k-1}[\,k > j\,]$. Show that the measures $\mu^x_{(t_1,\dots,t_n)}$ are consistent, i.e.

i) show that

$$\mu^x_{(t_1,\dots,t_n)}(B_{1,j} \times \dots \times B_{n,j}) = \mu^x_{\pi_{j,n}(t_1,\dots,t_n)}(C_{1,j} \times \dots \times C_{n-1,j})$$

for every $n \in \mathbb{N}$, for every $1 \leq j \leq n$.

Suggestion. There is an unfortunate misprint s_k should be $s_k = t_k[\,k < j\,]+t_{k+1}[\,k \geq j\,]$ and similarly $C_{k,j}$ should have $[\,A_{k+1}\,][\,k \geq j\,]$. Furthermore, the n should be at least 2 for this to be meaningful.

Therefore, this problem is considered to solved by everyone.

With this change let's look for different values of n. First is $n = 2$. Then we have two equations

$$\mu^x_{(t_1,t_2)}(A_1 \times S) = \mu^x_{(t_1)}(A_1).$$

and

$$\mu^x_{(t_1,t_2)}(S \times A_2) = \mu^x_{(t_2)}(A_2).$$

Therefore, by definition these reduce to equations

$$\int_{A_1} P_{t_2-t_1,x_1}(S) P_{t_1,x}(\,\mathrm{d}x_1) = P_{t_1,x}(A_1)$$

and

$$\int_S P_{t_2-t_1,x_1}(A_2)P_{t_1,x}(\,\mathrm{d}x_1) = P_{t_2,x}(A_2).$$

The first one follows since $P_{t,x}$ is a probability measure for every t and x and hence $P_{t_2-t_1,x_1}(S) = 1$ and thus

$$\int_{A_1} P_{t_2-t_1,x_1}(S)P_{t_1,x}(\,\mathrm{d}x_1) = \int_{A_1} P_{t_1,x}(\,\mathrm{d}x_1) = P_{t_1,x}(A_1).$$

The second one is the Chapman–Kolmogorov equation (since $t_2 - t_1 + t_1 = t_2$) so it holds as well. Let's suppose that the claim holds for all n (say less than N) and let's consider the case $n + 1$. So we have $n + 1$ equations to show. Suppose $k = n + 1$ first. Then

$$\begin{aligned}\mu^{(x)}_{(t_1,t_2,\dots,t_{n+1})}(A_1 \times A_2 \times \dots \times S) &= \int_{A_1} \dots \int_{A_n} P_{t_1,x}(\,\mathrm{d}x_1) \dots P_{t'_n,x_{n-1}}(A_n)P_{t'_{n+1},x_n}(S) \\ &= \mu^{(x)}_{(t_1,\dots,t_n)}(A_1 \times \dots \times A_n)\end{aligned}$$

as in the case when $n = 2$. Next suppose $1 < k \le n$. Since the equation is then of form

$$\int_{A_1} \dots \int_{A_{k-1}} P_{t'_{k-1}}(\,\mathrm{d}x_{k-1})f(x_{k-1}) = \int_{A_1} \dots \int_{A_{k-1}} P_{t'_{k-1}}(\,\mathrm{d}x_{k-1})g(x_{k-1})$$

it is enough to show that $f = g$. Now we notice that

$$f(x_{k-1}) = \mu^{(x_{k-1})}_{(t'_k,v_{k+1},t'_{k+2},\dots,t'_{n+1})}(S \times A_{k+1} \times \dots \times A_{n+1})$$

and

$$g(x_{k-1}) = \mu^{(x_{k-1})}_{(v_{k+1},t'_{k+2},\dots,t'_{n+1})}(A_{k+1} \times \dots \times A_{n+1})$$

where $v_{k+1} = t_{k+1} - t_{k-1}$. Since now there are at most $n+1-1$ times and sets left, we get that $f = g$ by induction assumption.

So we are left with the equation for $k = 1$. This time we notice that the equation is of form

$$\int_S P_{t_1,x}(\,\mathrm{d}x_1) \int_{A_2} P_{t'_2,x}(\,\mathrm{d}x_2)f(x_2) = \int_{A_2} P_{t_2,x}(\,\mathrm{d}x_2)f(x_2)$$

Notice that if f would be a constant function, this would be the Chapman–Kolmogorov equation. Now Chapman–Kolmogorov equation implies for simple functions f that

$$\int_S P_{t_1,x}(\,\mathrm{d}x_1) \int_{A_2} P_{t'_2,x}(\,\mathrm{d}x_2)f(x_2) = \int_{A_2} P_{t_2,x}(\,\mathrm{d}x_2)f(x_2)$$

holds. And from this the claim follows by the monotone convergence.

2. Show that the transition probability operator of Brownian motion $P_t^{(B)}$ (see pages 21–22) satisfies

$$\lim_{t\to 0} P_t^{(B)}\ f(x) = f(x)$$

for every $x \in \mathbb{R}^d$ and every bounded and continuous f. (Hint: change of variables so that t appears only in the argument of f and dominated convergence).

Suggestion. By definition

$$P_t^{(B)} f(x) = (2\pi t)^{-d/2} \int_{\mathbb{R}^d} f(y) e^{-\frac{1}{2}|x-y|^2/t}\,\mathrm{d}y.$$

By change of variable $z = (y-x)/\sqrt{t}$ we see that $f(y) = f(x + z\sqrt{t})$, the exponent $|x-y|^2/t = |z|^2$ and the $\mathrm{d}z = t^{-d/2}\,\mathrm{d}y$. So

$$P_t^{(B)} f(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x + z\sqrt{t}) e^{-\frac{1}{2}|z|^2}\,\mathrm{d}z.$$

Since f is bounded, and $e^{-\frac{1}{2}|z|^2}$ is integrable, we have by dominated convergence that

$$\begin{aligned}\lim_{t\to 0} P_t^{(B)} f(x) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \lim_{t\to 0} f(x + z\sqrt{t}) e^{-\frac{1}{2}|z|^2}\,\mathrm{d}z \\ &= f(x)(2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}|z|^2}\,\mathrm{d}z.\end{aligned}$$

Now the integral on the right is the Gaussian integral and so the right-hand side is 1 and the claim follows.

3. Show that the transition probability operator of Brownian motion $P_t^{(B)}$ (see 2. above) satisfies

$$\lim_{t\to 0} \| P_t^{(B)} f - f \| = \lim_{t\to 0} \sup_{x\in\mathbb{R}^d} |P_t^{(B)} f(x) - f(x)| = 0$$

for every $x \in \mathbb{R}^d$ and every $f \in C_\infty(\mathbb{R}^d)$. (Hint: show that there is large R such that outside ball of radius R the difference is small and 2.)

Suggestion. Let $\varepsilon > 0$. Choose $R_0 > 0$ so that $|f(x)| < \varepsilon$ for every $|x| > R_0$. So let's take any point x, such that $|x| > R > R_0$ and let's divide the integration in to two parts

$$|P_t^{(B)} f(x)| \leq (2\pi t)^{-d/2} \int_{|y|\leq r} \mathrm{d}y \cdots + (2\pi t)^{-d/2} \int_{|y|>r} \mathrm{d}y \ldots$$

Since f is bounded, say $|f(y)| \leq M$ for every $y \in \mathbb{R}^d$, we can estimate the first one by using polar coordinates to be at most

$$M(2\pi t)^{-d/2} \int_0^r e^{-(\lambda-|x|)^2/2} \lambda^{d-1} \, d\lambda.$$

If $r < R/2$, say, then $|\lambda - |x|| = |x| - \lambda > R/2$ and hence $-(\lambda - |x|)^2 \leq -R^2/4$. So we may estimate the first integral by

$$M' e^{-R^2/32} r^d \leq M'' e^{-R^2/8} R^d$$

Now the second integral is easy to estimate, if we assume that $r > R_0$, since then $|f(y)| < \varepsilon$ for every $|y| > r$ and the remaining part is just the Gaussian integral. So if we can choose $r < R/2$ and $r > R_0$, then

$$|P_t^{(B)} f(x)| \lesssim \varepsilon + e^{-R^2/8} R^d.$$

This means that the estimate can be done provided that $R_0 < r < R/2$ which means that the estimate holds whenever $R > 2R_0$. Moreover, we can make this estimate smaller than 2ε for large enough R. Let's call such an R as R_1.

Now for R_1 we have

$$|P_t^{(B)} f(x) - f(x)| \leq 2\varepsilon + |f(x)| \leq 3\varepsilon$$

for every $|x| > R_1$. Since the set $|x| \leq R_1$ is bounded and closed, i.e. compact, we know that the pointwise convergence implies uniform convergence so that

$$\lim_{t \to 0} \sup_{|x| \leq R_1} |P_t^{(B)} f(x) - f(x)| = 0$$

if

$$\lim_{t \to 0} |P_t^{(B)} f(x) - f(x)| = 0$$

for every $|x| \leq R_1$. But this we know from the Problem 2. Therefore, if we choose $t_0 > 0$ small enough, we have that

$$\sup_{|x| \leq R_1} |P_t^{(B)} f(x) - f(x)| \leq 3\varepsilon$$

for every $t < t_0$ and so

$$\sup_{x \in \mathbb{R}^d} |P_t^{(B)} f(x) - f(x)| \leq 3\varepsilon$$

for every $t < t_0$. This implies the claim.

Some theory. Suppose (X_t) is a Markov process and suppose there exists a *shift operators* $\theta_s\colon \Omega \to \Omega$ for every $s > 0$ with the property $X_t \circ \theta_s(\omega) = X_t(\theta_s(\omega)) := X_{t+s}(\omega)$ for every $t \geq 0$ and for every $\omega \in \Omega$. This way we can easily express that general idea of *forgetting the past* and the *restarting the clock.*
Note. In Problems 4, 5, 6 and 7 the triplet $(X_t, \mathscr{F}_t, \{\mathbf{P}_x\})$ is a Markov process. The algebra $\mathscr{A}_\infty$ and the σ-algebre $\mathscr{F}_\infty$ are defined in the proof of Lemma 3.14 on page 24. The Dynkin system is defined in the proof of Lemma 3.14 on page 25.

4. Let $s < t_1 < \cdots < t_n$. Show that the time stationary Markov property for (X_t) with respect to $(\mathscr{F}_t)$ implies that

$$\mathbf{E}_x(f_1(X_{t_1}) \dots f_n(X_{t_n}) \,|\, \mathscr{F}_s) = \mathbf{E}_{X_s}\,(f_1(X_{t_1-s}) \dots f_n(X_{t_n-s}))$$

for every $x \in S$ and for every f_j bounded and measurable function with $j = 1, \dots, n$. You can assume that this is know for $n = 1$.
Suggestion. When $n = 1$ this claim states that

$$\mathbf{E}_x(f_1(X_{t_1}) \,|\, \mathscr{F}_s) = \mathbf{E}_{X_s}\, f_1(X_{t_1-s})$$

This is the Markov property in the integrated form since suppose f_1 is a simple function $a_1[\,B_1\,] + \cdots + a_n[\,B_n\,]$. Then

$$\begin{aligned}\mathbf{E}_x(f_1(X_{t_1}) \,|\, \mathscr{F}_s) &= \sum_k a_k \mathbf{E}_x([\,X_{t_1} \in B_k\,] \,|\, \mathscr{F}_s) = \sum_k a_k \mathbf{E}_{X_s}\,[\,X_{t_1-s} \in B_k\,] \\ &= \mathbf{E}_{X_s}\, f_1(X_{t_1-s})\end{aligned}$$

Therefore, by the usual procedure we can use monotone convergence to show this for every bounded and measurable function f_1. But this part was something you could assume to be known.

Assume that the claim holds for $N \leq n$ and let's consider the case $N = n + 1$. Since $f_1(X_{t_1})$ is $\mathscr{F}_{t_1}$-measurables, we have by the property of the conditional expectation that

$$\mathbf{E}_x(f_1(X_{t_1}) \dots f_n(X_{t_n}) \,|\, \mathscr{F}_s) = \mathbf{E}_x(f_1(X_{t_1}) \mathbf{E}_x(f_2(X_{t_2}) \dots f_n(X_{t_n}) \,|\, \mathscr{F}_{t_1}) \;|\, \mathscr{F}_s)\;.$$

If you don't remember this I'll verify this in the end. But now we may use the induction assumption and we have that

$$\mathbf{E}_x(f_2(X_{t_2}) \dots f_n(X_{t_n}) \,|\, \mathscr{F}_{t_1}) = \mathbf{E}_{X_{t_1}}\, f_2(X_{t_2-t_1}) \dots f_n(X_{t_n-t_1}) = g(X_{t_1}).$$

for every x. So we have obtained that

$$\mathbf{E}_x(f_1(X_{t_1})\dots f_n(X_{t_n})\,|\,\mathscr{F}_s) \;= \mathbf{E}_x(f_1(X_{t_1})g(X_{t_1})\,|\,\mathscr{F}_s) \;= \mathbf{E}_{X_s}\,f_1(X_{t_1-s})g(X_{t_1-s})$$

where the last identity was shown first (or assumed to be known). Now recall

$$g(x) = \mathbf{E}_x\,f_2(X_{t_2-t_1})\dots f_n(X_{t_n-t_1})$$

and so again by the induction assumption

$$g(X_{t_1-s}) = \mathbf{E}_{X_{t_1-s}}\,f_2(X_{t_2-t_1})\dots f_n(X_{t_n-t_1}) \;= \mathbf{E}_x(f_2(X_{t_2-s})\dots f_n(X_{t_n-s})\,|\,\mathscr{F}_{t_1-s})$$

for every x. Especially,

$$\begin{aligned}\mathbf{E}_x(f_1(X_{t_1})\dots f_n(X_{t_n})\,|\,\mathscr{F}_s) \;&= \mathbf{E}_{X_s}\,f_1(X_{t_1-s})\mathbf{E}_{X_s}(f_2(X_{t_2-s})\dots f_n(X_{t_n-s})\,|\,\mathscr{F}_{t_1-s})\\ &= \mathbf{E}_{X_s}\,f_1(X_{t_1-s})f_2(X_{t_2-s})\dots f_n(X_{t_n-s})\end{aligned}$$

where the latter identity follows again from the fact that $f_1(X_{t_1-s})$ is $\mathscr{F}_{t_1-s}$-measurable by the same property of conditional expectation we used once already.

We were using two properties of conditional expectation. One is $\mathbf{E}\,(XY\,|\,\mathscr{G}) = X\mathbf{E}\,(Y\,|\,\mathscr{G})$ when X is $\mathscr{G}$-measurable and the integration makes sense. This was one of the Problems in the Excercise Sheet 1. And the second is $\mathbf{E}\,(X\,|\,\mathscr{G}) = \mathbf{E}\,(\mathbf{E}\,(X\,|\,\mathscr{H})\,\,|\,\mathscr{G})$ for every sub-σ-algebra $\mathscr{G}\subset\mathscr{H}$. This follows since this is equivalent with

$$\mathbf{E}\,[\,A\,]X \;= \mathbf{E}\,[\,A\,]\mathbf{E}\,(X\,|\,\mathscr{H})$$

for every $A\in\mathscr{G}$. When $A\in\mathscr{G}\subset\mathscr{H}$, we see that this holds by the definition of the conditional expectation with respect to $\mathscr{H}$.

5. Let $0<t_1<\dots<t_n$ and let $A=\{X_{t_1}\in A_1,\dots,X_{t_n}\in A_n\}\in\mathscr{A}_\infty$. Show that the time stationary Markov property for (X_t) with respect to $(\mathscr{F}_t)$ implies that

$$\mathbf{E}_x([\,A\,]\;\circ\theta_s\,|\,\mathscr{F}_s) \;= \mathbf{E}_{X_s}\,[\,A\,] \tag{0.1}$$

for every $x\in S$. (Hint: try writing $[\,A\,]\circ\theta_s$ as a form of Problem 4. from the above definition of the shift operator)

Suggestion. Note that

$$[\,A\,](\omega) = [\,X_{t_1}(\omega)\in A_1,\dots X_{t_n}(\omega)\in A_n\,]$$

and that

$$
\begin{aligned}
[\,A\,] \circ \theta_s(\omega) &= [\,X_{t_1}(\theta_s(\omega)) \in A_1, \dots X_{t_n}(\theta_s(\omega)) \in A_n\,] \\
&= [\,X_{t_1+s}(\omega) \in A_1, \dots X_{t_n+s}(\omega) \in A_n\,]
\end{aligned}
$$

This means that the claim is equivalent with

$$
\mathbf{P}_x\,(\,X_{t_1+s} \in A_1, \dots X_{t_n+s} \in A_n \,|\, \mathscr{F}_s\,) = \mathbf{P}_{X_s}\,(\,X_{t_1} \in A_1, \dots X_{t_n} \in A_n\,)
$$

If we denote $t'_j = t_j + s$ and denote $f_j = [\,A_j\,]$ the claim becomes

$$
\mathbf{E}_x\Big(f_1(X_{t'_1}) \dots f_n(X_{t'_n}) \,|\, \mathscr{F}_s\Big) \;= \mathbf{E}_{X_s}\,(f_1(X_{t'_1-s}) \dots f_n(X_{t'_n-s}))
$$

which holds by Problem 4.

6. Let $\mathscr{G}$ be the family of sets

$$
\mathscr{G} = \{\; A \in \mathscr{F}_\infty : \;\; A \text{ satifies the identity (0.1) }\}
$$

Show that $\mathscr{G}$ is a Dynkin system and that it coincides with $\mathscr{F}_\infty$. (Hint: see proof of Lemma 3.14).

Suggestion. We need to verify that

i) $\Omega \in \mathscr{G}$,

ii) $A^C \in \mathscr{G}$, if $A \in \mathscr{G}$,

ii) $\bigcup_k A_k \in \mathscr{G}$, if $\forall$: $A_k \in \mathscr{G}$ and $A_k \cap A_j = \emptyset$ for $j \neq k$.

The first one follows, since $\Omega \in \mathscr{A}_\infty$ and we proved that in Problem 5. The second claim is that

$$
\mathbf{E}_x\Big([\,A^C\,] \;\circ \theta_s \,|\, \mathscr{F}_s\Big) \;= \mathbf{E}_{X_s}\,[\,A^C\,]\,.
$$

if $A \in \mathscr{G}$. Since $[\,A^C\,] = 1 - [\,A\,]$ the claim becomes

$$
1 - \mathbf{E}_x([\,A\,] \;\circ \theta_s \,|\, \mathscr{F}_s) \;= 1 - \mathbf{E}_{X_s}\,[\,A\,]
$$

which is equivalent with $A \in \mathscr{G}$.

The third case is similar, since for disjoint sequence of sets

$$
[\bigcup_k A_k\,] \;= \sum_k [\,A_k\,]
$$

so the claim of *iii*) becomes

$$\sum_k \mathbf{E}_x([\, A_k \,] \ \circ \theta_s \,|\, \mathscr{F}_s) \ = \sum_k \mathbf{E}_{X_s} [\, A_k \,]$$

when $(A_k) \subset \mathscr{G}$ is sequence of disjoint sets. Since $A_k \in \mathscr{G}$ means that

$$\mathbf{E}_x([\, A_k \,] \ \circ \theta_s \,|\, \mathscr{F}_s) \ = \mathbf{E}_{X_s} [\, A_k \,]$$

the claim follows. Therefore, $\mathscr{G}$ is a Dynkin system. By the proof of Lemma 3.14. we know that $\sigma(\mathscr{A}_\infty) = \mathscr{F}_\infty$ and by Dynkin's π-λ Theorem, $\mathscr{G} \supset \sigma(\mathscr{A}_\infty) = \mathscr{F}_\infty$. But since $\mathscr{G} \subset \mathscr{F}_\infty$, we have that $\mathscr{G} = \mathscr{F}_\infty$ and the claim follows.

7. Let Z be a $\mathscr{F}_\infty$-measurable bounded random variable. Show that the time stationary Markov property for (X_t) with respect to $(\mathscr{F}_t)$ implies that

$$\mathbf{E}_x(Z \circ \theta_s \,|\, \mathscr{F}_s) \ = \mathbf{E}_{X_s} Z$$

for every $s \geq 0$. (Hint: first simple Z, then the general case).
Suggestion. When $Z = a_1[\, A_1 \,] \ + \ldots a_n[\, A_n \,]$ for $A_j \in \mathscr{F}_\infty$, we know by Problem 6. that $A_j \in \mathscr{G}$ and therefore,

$$\mathbf{E}_x([\, A_k \,] \ \circ \theta_s \,|\, \mathscr{F}_s) \ = \mathbf{E}_{X_s} [\, A_k \,]$$

This implies that

$$\mathbf{E}_x(Z \ \circ \theta_s \,|\, \mathscr{F}_s) \ = \sum_k a_k \mathbf{E}_x([\, A_k \,] \ \circ \theta_s \,|\, \mathscr{F}_s) \ = \sum_k a_k \mathbf{E}_{X_s} [\, A_k \,] \ = \mathbf{E}_{X_s} Z$$

The general case follows by monotone convergence.

8. Suppose Z is an integrable real valued random variable and $(\mathscr{F}_t)$ is a filtration. Show that a process (X_t) which is *defined* as

$$X_t := \mathbf{E} \ (Z \,|\, \mathscr{F}_t)$$

is a martingale with respect to the filtration $(\mathscr{F}_t)$.
Suggestion. We first need to check that X is adapted, $\mathbf{E} \,|\, |X_t < \infty$ for every t and then the martingale property.

Since X_t is conditional expectation with respect to the filtration $(\mathscr{F}_t)$, it is by the definition of conditional expectation measurable $\mathscr{F}_t$-measurable. Thus, X is adapted.

Next, since Z is integrable, we can write $|\mathbf{E}\,(Z\,|\,\mathscr{F}_t)\,| \leq \mathbf{E}\,(|Z|\,|\,\mathscr{F}_t)$ and therefore,

$$\mathbf{E}\,|X_t| \;\leq \mathbf{E}\,\mathbf{E}\,(|Z|\,|\,\mathscr{F}_t) \;= \mathbf{E}\,|Z| \;< \infty,$$

and hence X_t is integrable for every t as well.

The martingale property follows from the property we have used couple of times (see Problem 4.). By definition of X_t we have

$$\mathbf{E}\,(X_t\,|\,\mathscr{F}_s) = \mathbf{E}\,(\mathbf{E}\,(Z\,|\,\mathscr{F}_t)\;|\,\mathscr{F}_s)\;.$$

Now since $\mathscr{F}_s \subset \mathscr{F}_t$, we have that

$$\mathbf{E}\,(\mathbf{E}\,(Z\,|\,\mathscr{F}_t)\;|\,\mathscr{F}_s) = \mathbf{E}\,(Z\,|\,\mathscr{F}_s) = X_s$$

and the claim follows. This is a very important class of martingales, actually by Theorem 5.3. every uniformly integrable martingale is of this type and we see from this Problem that we can find a lot of martingales given a filtration.

9. Let τ_1 and τ_2 be $(\mathscr{F}_t)$-stopping times. Show that

$$\tau_1 \wedge \tau_2, \quad \tau_1 \vee \tau_2, \quad \text{and} \quad \tau_1 + \tau_2$$

are $(\mathscr{F}_t)$-stopping times. (Hint: try to express the conditions as unions and intersections of conditions involving only τ_1 and τ_2. Also discrete time versions are fine.)
Suggestion.

We will show first that $\{\tau_1 \wedge \tau_2 \leq t\} \in \mathscr{F}_t$. If $\tau_1 \wedge \tau_2 \leq t$, then either $\tau_1 \leq t$ or $\tau_2 \leq t$ (or both). Moreover, if $\tau_1 \leq t$ or $\tau_2 \leq t$, then the minimum is also $\leq t$, so

$$\{\tau_1 \wedge \tau_2 \leq t\} = \{\tau_1 \leq t\} \cup \{\tau_2 \leq t\} \in \mathscr{F}_t$$

since τ_1 and τ_2 are stopping times and therefore both of the events on the right-hand side and thus also their union is in $\mathscr{F}_t$.

We will show first that $\{\tau_1 \vee \tau_2 \leq t\} \in \mathscr{F}_t$. If $\tau_1 \vee \tau_2 \leq t$, then both $\tau_1 \leq t$ and $\tau_2 \leq t$. Moreover, if both $\tau_1 \leq t$ and $\tau_2 \leq t$, then the maximum is also $\leq t$, so

$$\{\tau_1 \vee \tau_2 \leq t\} = \{\tau_1 \leq t\} \cap \{\tau_2 \leq t\} \in \mathscr{F}_t$$

since τ_1 and τ_2 are stopping times and therefore both of the events on the right-hand side and thus also their union is in $\mathscr{F}_t$.

Let's do the discrete time version first and then the general case.

Now $\tau_1 \in \{t_1, \dots, t_n\}$ and $\tau_2 \in \{s_1, \dots, s_m\}$ for some $0 \le t_1 < \cdots < t_n$ and $0 \le s_1 < \cdots < s_m$.

Now $\tau_1 + \tau_2 \in \{\ t_j + s_k :\ j, k\ \}$ and $\tau_1 + \tau_2 \le t$, when ever $\tau_1 = t_j$ and $\tau_2 = s_k$ satisfy the extra conditions $0 \le t_j \le t$ following $0 \le s_k \le t - t_j$. So

$$\{\tau_1 + \tau_2 \le t\} = \bigcup_{t_j \le t} \bigcup_{s_k \le t - t_j} \{\tau_1 = t_j, \tau_2 = s_k\}$$

For every fixed $t_j \le t$ and $s_k \le t - t_j$ the event on the right is in $\mathscr{F}_u \subset \mathscr{F}_t$, where $u = t_j \vee s_k \le t$. So everything on the right is in $\mathscr{F}_t$ so the claim follows in the discrete case for simple stopping times.

We need to reformulate the right-hand side for the general case since we cannot expect to have equalities, since only inequalities generalise. First we take the union over s_k for fixed t_j and we notice that

$$\{\tau_1 + \tau_2 \le t\} = \bigcup_{t_j \le t} \{\tau_1 = t_j, \tau_2 \le t - t_j\}$$

So now τ_2 is fine. Next we notice that if $\tau_2 \le t - t_j$, then any $\tau_1 = t_l$ for $t_l \le t_j$ must actually appear on the right, since $\tau_2 + \tau_1 \le t - t_j + t_l < t$. So we can write the above as

$$\{\tau_1 + \tau_2 \le t\} = \bigcup_{t_j \le t} \{\tau_1 \le t_j, \tau_2 \le t - t_j\}$$

This is a good formulation for generalization and first we good try

$$\{\tau_1 + \tau_2 \le t\} = \bigcup_{s \le t} \{\tau_1 \le s, \tau_2 \le t - s\}$$

We notice that both sides are equal and for every fixed $s \le t$ the event on the right-hand side is in $\mathscr{F}_t$, but there is one issue left and that's the uncountability of the union.

But, fortunately, this is almost easy to fix, namely by restricting $s \in \mathbb{Q}$. Then

$$\{\tau_1 + \tau_2 \le t\} = \bigcup_{s < t, s \in \mathbb{Q}} \{\tau_1 \le s, \tau_2 \le t - s\} \cup \{\tau_1 = t, \tau_2 = 0\}$$

holds as well, since now the right-hand side is by previous a subset of $\{\tau_1 + \tau_2 \le t\}$, but if $\tau_1(\omega) + \tau_2(\omega) \le t$, then either $\tau_1(\omega) = t$ or $\tau_1(\omega) < t$. The former case is the special case on the right and in the latter case we can find a rational s such that $\tau_1(\omega) \le s < t$ and $\tau_2(\omega) + s < t$.

Why? Well the first condition says that $s \in [\tau_1(\omega), t)$ and the latter says that $s \in (-\infty, t - \tau_2(\omega))$. These intervals intersect if and only if $t - \tau_2(\omega) > \tau_1(\omega)$ which is equivalent with $\tau_1(\omega) + \tau_2(\omega) < t$.

Therefore, we have

$$\{\tau_1 + \tau_2 \leq t\} = \bigcup_{s<t, s\in\mathbb{Q}} \{\tau_1 \leq s, \tau_2 \leq t - s\} \cup \{\tau_1 = t, \tau_2 = 0\}$$

and the right-hand side is a countable union of events in $\mathscr{F}_t$.

10. Show that the $\mathscr{F}_\tau$ is a σ-algebra, when τ is a $(\mathscr{F}_t)$-stopping time.
Suggestion. Now by definition $\mathscr{F}_\tau = \{\ A : \ \forall t \geq 0 \colon A \cap \{\tau \leq t\} \in \mathscr{F}_t\ \}$. We prove this by verifyng that $\mathscr{F}_\tau$ satisfies the axioms of the σ-algebra.

i) $\Omega \in \mathscr{F}_\tau$, since $\Omega \cap \{\tau \leq t\} \ = \{\tau \leq t\}$ and this is in $\mathscr{F}_t$ since τ is a proper stopping time.

ii) Suppose $A \in \mathscr{F}_\tau$. Then

$$\{A^C, \tau \leq t\} = \{\tau \leq t\} \setminus \{A, \tau \leq t\} = \{\Omega, \tau \leq t\} \cap \{A, \tau \leq t\}^C.$$

Now the events on the right-hand side are in $\mathscr{F}_t$ since $\Omega, A \in \mathscr{F}_\tau$ so we deduce that $A^C \in \mathscr{F}_\tau$.

iii) Suppose $(A_k) \subset \mathscr{F}_\tau$. Since

$$\{\tau \leq t\} \ \cap \bigcup_k A_k = \bigcup_k \{A_k, \tau \leq t\}$$

and by assumption everything on the right is in $\mathscr{F}_t$ we have that $\bigcup A_k \in \mathscr{F}_\tau$ as well.

11. Let $(\mathscr{F}_t)$ be a filtration. Show that the $(\mathscr{F}_{t+})$ is right-continuous filtration.
Suggestion. To show that $(\mathscr{G}_t) = (\mathscr{F}_{t+})$ is right-continuous filtration, we need to show that

$$\bigcap_{s>t} \mathscr{G}_s = \mathscr{G}_t.$$

Clearly, $\mathscr{G}_t \subset \bigcap \mathscr{G}_s$ if we know that $(\mathscr{G}_t)$ is a filtration. So let's suppose $A \in \bigcap \mathscr{G}_s$. By definition, $A \in \mathscr{F}_{s+}$ for every $s > t$. Which by definition means that $A \in \mathscr{F}_u$ for every $u > s$ and every $s > t$. But this implies that $A \in \mathscr{F}_u$ for every $u > t$ and therefore, $A \in \mathscr{F}_{t+} = \mathscr{G}_t$. So, at least if we know that $(\mathscr{G}_t)$ is a filtration, then it is right-continuous.

For completeness, let's show that $\mathscr{G}_t$ is a σ-algebra.

i) The $\Omega \in \mathscr{G}_t$ is quite easy to verify, since $\Omega \in \mathscr{F}_s$ for every s, and therefore, also in every $\mathscr{F}_s$ for $s > t$.

ii) Next, if $A \in \mathscr{G}_t$, then $A \in \mathscr{F}_s$ for every $s > t$. Therefore, $A^C \in \mathscr{F}_s$ for every $s > t$ which means that $A^C \in \mathscr{G}_t$.

iii) Next, if $(A_k) \subset \mathscr{G}_t$, then $(A_k) \subset \mathscr{F}_s$ for every $s > t$. Therefore, $A = \bigcup A_k \in \mathscr{F}_s$ for every $s > t$ which means that $A \in \mathscr{G}_t$.

Let's still verify that $\mathscr{G}_t \subset \mathscr{G}_u$ for every $t < u$. If $A \in \mathscr{G}_t$, then as above $A \in \mathscr{F}_s$ for every $s > t$. Especially, $A \in \mathscr{F}_u \subset \mathscr{G}_u$.

12. Show that a random variable τ is a $(\mathscr{F}_{t+})$-stopping time if and only if for every $t > 0$ it holds that $\{\tau < t\} \in \mathscr{F}_t$. (Hint. $\Longrightarrow$ consider events $\{\tau \leq t - 1/k\}$ and $\Longleftarrow$ consider events $\{\tau < t + 1/k\}$.) **Suggestion**.

$\Longrightarrow$ If τ is $(\mathscr{F}_{t+})$-stopping time, then $\{\tau \leq s\} \in \mathscr{F}_{s+}$ for every s. Following the hint, let's take $s = t - \frac{1}{k}$. Then

$$\{\tau \leq t - \tfrac{1}{k}\} \in \mathscr{F}_{s+} \subset \mathscr{F}_t$$

for every $k > 0$. The second part $\mathscr{F}_{s+} \subset \mathscr{F}_t$ follows, since $t > t - \frac{1}{k} = s$ and hence every $A \in \mathscr{F}_{s+}$ must belong to $\mathscr{F}_t$ as well. Moreover,

$$\{\tau < t\} = \bigcup_k \{\tau \leq t - \tfrac{1}{k}\}$$

so we have deduced that $\{\tau < t\} \in \mathscr{F}_t$ as claimed.

$\Longleftarrow$ Suppose now that $\{\tau < s\} \in \mathscr{F}_s$ for every $s > 0$. Following the hint, take $s_k = t + \frac{1}{k}$. This implies that

$$B_k := \{\tau < t + \tfrac{1}{k}\} \in \mathscr{F}_{s_k}.$$

Now

$$B := \{\tau \leq t\} = \bigcap_{k=1}^{\infty} \{\tau < t + \tfrac{1}{k}\} = \bigcap_{k=1}^{\infty} B_k,$$

so we need to show that $B \in \mathscr{F}_{t^+}$. Since $\{\tau < t + \frac{1}{k+1}\} \subset \{\tau < t + \frac{1}{k}\}$, we see that (B_k) is *decreasing* sequence of events. This implies that

$$B = \bigcap_{k=1}^{\infty} B_k = \bigcap_{k=M}^{\infty} B_k$$

for every $M \geq 1$, since the clearly the left-hand side is a subset of the right-hand side, but if $\omega \in B_k$ for every $k \geq M$, then it also belongs to $\omega \in B_M \subset B_{M-1} \subset \cdots \subset B_1$.

So let's assume that $k \geq M$. Then $s_k = t + \frac{1}{k} \leq t + \frac{1}{M} = s_M$, and so $B_k \in \mathscr{F}_{s_k} \subset \mathscr{F}_{s_M}$. This implies that

$$B = \bigcap_{k=M}^{\infty} B_k \in \mathscr{F}_{s_M}$$

for every $M \geq 1$. If $s > t$, then we can find $M \geq 1$ so that $s > s_M > t$ and so, $B \in \mathscr{F}_s$ as well. But this means that

$$B \in \mathscr{F}_s$$

for every $s > t$ or in other words $B = \{\tau \leq t\} \in \mathscr{F}_{t^+}$. This proves the claim.