Department of Mathematics and Statistics
Stochastic processes on domains
Suggestions to excercise problem sheet 1

1. Let $\Omega = [0,1)$ be the unit interval and denote $I_j = [j2^{-n},(j+1)2^{-n})$ for $j \in \mathbb{Z}$ and $n \in \mathbb{N}_+$. Show that the finite family $\mathscr{G}_n := \{\ I_{j_1} \cup \dots I_{j_m} :\ 0 \le j_1 < \dots j_m < 2^n\ \}$ is a σ-algebra on Ω.

Suggestion. We need to check that $\mathscr{G}_n$ contains Ω, but since $\Omega = I_0 \cup \dots \cup I_{2^n-1}$ it is in $\mathscr{G}_n$.

Next if $A \in \mathscr{G}_n$, then $A = I_{j_1} \cup \dots \cup I_{j_k}$ for some $0 \le j_1 < \dots j_k \le 2^{n-1}$. Now $\{0,\dots,2^n-1\} \setminus \{j_1,\dots,j_k\} = \{i_1,\dots,i_{2^n-1-k}\}$ and therefore, $A^C = I_{i_1} \cup \dots \cup I_{I_{2^n-1-k}} \in \mathscr{G}_n$.

Now since $\mathscr{G}_n$ is finite, we only need to check that the finite unions are in $\mathscr{G}_n$ since $\{A_1,\dots,A_j,\dots\} = \{B_1,\dots,B_M\}$ for some $B_j \in \mathscr{G}_n$. Let $A = B_1 \cup \dots \cup B_M$ where

$$B_j = \bigcup_{k=1}^{m_j} I_{l_{k,j}}$$

for $0 \le l_{1,j} < \dots l_{m_j,j} < 2^n$. Then

$$A = \bigcup_{j=1}^{M} B_j = \bigcup_{j=1}^{M} \bigcup_{k=1}^{m_j} I_{l_{k,j}} \in \mathscr{G}_n.$$

2. Let $\Omega = [0,1)$ and $\mathscr{G}_n$ as in 1. Let $\mathscr{F} = \mathscr{B}[0,1)$, $\mathbf{P}$ be the Lebesgue measure on $(\Omega,\mathscr{F})$ and $\xi : \Omega \to \mathbb{R}_+$ be a Borel measurable. Show from the definition of conditional expectation that

$$\mathbf{E}\ (\xi \,|\, \mathscr{G}_n)\ (\omega) = \sum_{j=0}^{2^n-1} [\,\omega \in I_j\,] 2^n \int_{I_j} \xi(\omega')\,\mathrm{d}\omega'$$

Suggestion. First we note that the random variable

$$Z = \sum_{j=0}^{2^n-1} 2^n c_j [\,I_j\,]$$

where $c_j = \mathbf{E}\,\xi[\,I_j\,]$ on the right-hand side is integrable since it is bounded. Then we note that it is $\mathscr{G}_n$-measurable, since it takes finitely many values $\{2^n c_0, \dots, 2^n c_{2^n-1}\}$ and moreover,

$$\{Z = 2^n c_j\} = I_j \in \mathscr{G}_n.$$

We still need to verify that

$$\mathbf{E}\,[\,A\,]\xi = \mathbf{E}\,[\,A\,]Z\,.$$

Since every $A \in \mathscr{G}_n$ is of form $I_{j_1} \cup \cdots \cup I_{j_k}$ it is enough to show

$$\mathbf{E}\,[\,I_j\,]\xi = \mathbf{E}\,[\,I_j\,]Z \quad \forall j = 0, \dots, 2^n - 1.$$

The left-hand side is

$$\mathbf{E}\,[\,I_j\,]\xi = c_j$$

and the right-hand side is

$$\mathbf{E}\,[\,I_j\,]Z = \sum_{k=0}^{2^n-1} \mathbf{E}\,[\,I_j \cap I_k\,]2^n c_k = 2^n c_j \mathbf{E}\,[\,I_j\,] = 2^n 2^{-n} c_j = c_j.$$

Therefore, the right-hand side is equal to left-hand side for every j and the claim follows.

3. Let $f\colon \mathbb{R} \to \mathbb{R}_+$ be a simple, measurable function

$$f(x) = \sum_{k=1}^{n} a_k[\,x \in A_k\,].$$

Let Y be bounded and positive random variable, X a real-valued random variable and assume that X is $\mathscr{G}$-measurable for some sub-σ-algebra $\mathscr{G} \subset \mathscr{F}$. Show that the conditional expectation $f(X)Y$ with respect to $\mathscr{G}$ is

$$\mathbf{E}\,(f(X)Y \,|\, \mathscr{G}) = f(X)\mathbf{E}\,(Y \,|\, \mathscr{G})$$

almost surely.

Suggestion. Since $f(X)$ is bounded and Y is bounded, the product $f(X)Y$ is bounded and therefore integrable. Furthermore, since Y is bounded, it is integrable as well. So both $\mathbf{E}\,(f(X)Y \,|\, \mathscr{G})$ and $\mathbf{E}\,(Y \,|\, \mathscr{G})$ can be defined. Moreover, $f(X)\mathbf{E}\,(Y \,|\, \mathscr{G})$ is integrable and $\mathscr{G}$-measurable as a product of two $\mathscr{G}$-measurable functions, so the claim makes sense.

Next take any $A \in \mathscr{G}$. We want to show that

$$\mathbf{E}\,([\,A\,]f(X)Y) = \mathbf{E}\,([\,A\,]f(X)\mathbf{E}\,(Y\,|\,\mathscr{G}))$$

since then the claim follows. Since f is a simple function, this claim can be written as

$$\sum_{k=1}^{n} a_k \mathbf{E}\,([\,A, X \in A_k\,]Y) = \sum_{k=1}^{n} a_k \mathbf{E}\,([\,A, X \in A_k\,]\mathbf{E}\,(Y\,|\,\mathscr{G}))\,.$$

Now since X is $\mathscr{G}$-measurable, we know that $\{X \in A_k\} \in \mathscr{G}$ for every k, and therefore, $\{A, X \in A_k\} \in \mathscr{G}$ in every k. But the definition of the conditional expectation says that hence

$$\mathbf{E}\,([\,A, X \in A_k\,]\mathbf{E}\,(Y\,|\,\mathscr{G})) = \mathbf{E}\,([\,A, X \in A_k\,]Y)$$

for every k and the claim follows.

4. Let $f\colon \mathbb{R} \to \mathbb{R}_+$ be a bounded, positive and measurable function (but not necessarily simple) and otherwise assume the same as in 3. Show that the claim of 3. holds in this case as well by using monotone convergence theorem.

Suggestion. Since $f(X)$ is bounded and Y is bounded, the product $f(X)Y$ is bounded and therefore integrable. Furthermore, since Y is bounded, it is integrable as well. So both $\mathbf{E}\,(f(X)Y\,|\,\mathscr{G})$ and $\mathbf{E}\,(Y\,|\,\mathscr{G})$ can be defined. Moreover, $f(X)\mathbf{E}\,(Y\,|\,\mathscr{G})$ is integrable and $\mathscr{G}$-measurable as a product of two $\mathscr{G}$-measurable functions, so the claim makes sense.

Define for every $n > 0$ a simple function f_n such that

$$f_n(x) := \sum_{j=0}^{\infty} j2^{-n}[\,f(x) \in I_{j,n}\,]$$

where $I_{j,n} = [\,j2^{-n}, (j+1)2^{-n})$. Now $0 \le f_n(x) \le f(x)$ and[1] therefore, it is simple and measurable (since f is measurable and thus $\{\,x:\ f(x) \in I_{j,n}\,\}$ is measurable).

Moreover, $f_n(x) \le f_{n+1}(x)$ since $f_n(x)[\,f(x) \in I_{j,n}\,] = j2^{-n}$ and

$$\begin{aligned} f_{n+1}(x)[\,f(x) \in I_{j,n}\,] &= f_{n+1}(x)([\,f(x) \in I_{2j,n+1}\,] + [\,f(x) \in I_{2j+1,n+1}\,]) \\ &= 2j2^{-n-1}[\,f(x) \in I_{2j,n+1}\,] + (2j+1)2^{-n-1}[\,f(x) \in I_{2j+1,n+1}\,]) \\ &\ge j2^{-n}[\,f(x) \in I_{j,n}\,] \end{aligned}$$

[1] since $f_n(x)[\,f(x) \in I_{j,n}\,] = j2^{-n}[\,f(x) \in I_{j,n}\,] \le f(x)[\,f(x) \in I_{j,n}\,]$ for every j, and $\mathbb{R}_+ = \bigcup I_{j,n}$ and so $f(x) \in I_{j,n}$ for some j

So the sequence (f_n) is monotonically increasing. Also $f(x) - f_n(x) \leq 2^{-n}$ so $f = \lim_{n\to\infty} f_n$. Therefore, the monotone convergence theorem says that for every $A \in \mathscr{G}$

$$\mathbf{E}\,([\,A\,]f(X)Y) = \lim_{n\to\infty} \mathbf{E}\,([\,A\,]f_n(X)Y)\,.$$

By problem 3. we know that

$$\mathbf{E}\,([\,A\,]f_n(X)Y) = \mathbf{E}\,([\,A\,]f_n(X)\mathbf{E}\,(Y\,|\,\mathscr{G})\,)$$

Now the monotone convergence theorem says (again) that

$$\mathbf{E}\,([\,A\,]f(X)\mathbf{E}\,(Y\,|\,\mathscr{G})\,) = \lim_{n\to\infty} \mathbf{E}\,([\,A\,]f_n(X)\mathbf{E}\,(Y\,|\,\mathscr{G})\,)$$

Thus, we have shown that

$$\mathbf{E}\,([\,A\,]f(X)\mathbf{E}\,(Y\,|\,\mathscr{G})\,) = \mathbf{E}\,([\,A\,]f(X)Y)$$

for every $A \in \mathscr{G}$ and we are done.

A random variable $X = (X_1, \ldots, X_d)$ is a d-dimensional Gaussian random variable with *zero mean*, if its *characteristic function* $\varphi\colon \mathbb{R}^d \to \mathbb{C}$ is

$$\varphi_X(\lambda) := \mathbf{E}\,\exp\left(i\langle\,\lambda\,,\,X\,\rangle\right) = \exp(-\tfrac{1}{2}\langle\,\lambda\,,\,\Sigma\lambda\,\rangle)$$

for some positive definite symmetric matrix $\Sigma = (\mathbf{E}\,X_iX_j\,)_{ij} \in \mathbb{R}^{d\times d}$. Here $\langle\,x\,,\,y\,\rangle = x_1y_1 + \ldots x_dy_d$. A d-dimensional Gaussian random X variable with *zero mean* has a *density function* if the covariance matrix Σ (the matrix in 5.) is invertible. Then the density function is

$$q(x) = (2\pi)^{-d/2}|\Sigma|^{-1/2}\exp(-\tfrac{1}{2}\langle\,x\,,\,\Sigma^{-1}x)\,\rangle$$

where $|\Sigma|$ is the determinant of the matrix Σ. If $(X_1, X_2, \ldots, X_d)$ is a d-dimensional Gaussian random variable and $\mathbf{E}\,X_1X_j\, = 0$ for all $j \neq 1$, then X_1 is independent from $(X_2, \ldots, X_d)$.

5. Show the Lemma 2.2 (i) from Lecture notes (page 14).
Suggestion. We want to show that $X(t) = B(t+h) - B(h)$ is a Brownian motion. First let's check the expectation:

$$\mathbf{E}\,X(t)\, = \mathbf{E}\,B(t+h)\, - \mathbf{E}\,B(h)\, = 0 - 0 = 0.$$

Next let's compute the covariance. Let's assume that $0 \geq t < s$ and so $\min(t+h, s+h) = t+h$ and $\min(h, s+h) = \min(h, t+h) = h$ and $\min(h,h) = h$. Now

$$\begin{aligned}
\mathbf{E}\, X(t)X(s) &= \mathbf{E}\,(B(t+h) - B(h))(B(s+h) - B(h)) \\
&= \mathbf{E}\, B(t+h)B(s+h) \; + \mathbf{E}\, B(h)B(h) \\
&- \mathbf{E}\, B(t+h)B(h) \; - \mathbf{E}\, B(h)B(s+h) \\
&= (t+h) + h - h - h = t
\end{aligned}$$

so the covariance checks. We need to still show the Gaussianity, so let $\{t_1 < \cdots < t_n\}$ be time instances. We need to show that $Z = (X(t_1), \ldots, X(t_n))$ is a Gaussian random variable so let's compute its characteristic function

$$\varphi_Z(\lambda) = \mathbf{E}\, \exp\,(i\langle\, \lambda\,,\, Z\,\rangle) = \mathbf{E}\, \exp\,(i \sum_{j=1}^{n} \lambda_j (B(t_j + h) - B(h))).$$

Let's denote $Y = (B(t_1 + h), \ldots, B(t_n + h), B(h))$. This is Gaussian, since B is Brownian motion. If $\mu = (\lambda_1, \ldots, \lambda_n, -\sum_{j=1}^{n} \lambda_j)$, then we know that

$$\varphi_Y(\mu) = \exp(-\tfrac{1}{2}\langle\, \mu\,,\, \Sigma_Y \mu\,\rangle)$$

but by construction $\varphi_Z(\lambda) = \varphi_Y(\mu)$. So we only have to show that $\langle\, \mu\,,\, \Sigma_Y \mu\,\rangle = \langle\, \lambda\,,\, \Sigma_Z \lambda\,\rangle$. Let's denote $\Sigma_Z = A$ and $\Sigma_Y = C$.

By construction, $\mu_j = \lambda_j$ for every $j < n+1$ and $\mu_{n+1} = -\sum_{j=1}^{n} \lambda_j$. Moreover, by the previous computation

$$A_{j,k} = \mathbf{E}\, X(t_j)X(t_k) \; = t_{j \wedge k}$$

and when $j, k < n+1$ we have

$$C_{j,k} = \mathbf{E}\, B(t_j + h)B(t_k + h) \; = t_{j \wedge k} + h = A_{j,k} + h$$

In other cases $C_{j,k} = h$. Thus,

$$\sum_{j,k=1}^{n+1} \mu_j \mu_k C_{j,k} = \sum_{j,k=1}^{n} \lambda_j \lambda_k (A_{j,k} + h) + 2 \sum_{j=1}^{n} \lambda_j \mu_{n+1} h + \mu_{n+1}^2 h$$

This is tedious, but now notice that

$$\mu_{n+1}^2 = \sum_{j,k=1}^{n} \lambda_j \lambda_k$$

and

$$2\sum_{j=1}^{n}\lambda_j\mu_{n+1} = -2\sum_{j,k=1}^{n}\lambda_j\lambda_k = -2\mu_{n+1}^2,$$

and therefore,

$$\begin{aligned}\sum_{j,k=1}^{n+1}\mu_j\mu_k C_{j,k} &= \sum_{j,k=1}^{n}\lambda_j\lambda_k(A_{j,k}+h) + \sum_{j,k=1}^{n}\lambda_j\lambda_k h - 2\mu_{n+1}^2 h + \mu_{n+1}^2 h \\ &= \sum_{j,k=1}^{n}\lambda_j\lambda_k A_{j,k} + \sum_{j,k=1}^{n}\lambda_j\lambda_k h - \mu_{n+1}^2 h = \langle\, \lambda\,,\, A\lambda\,\rangle\end{aligned}$$

and the claim follows.

6. Show the Lemma 2.2 (ii) from Lecture notes (page 14).
Suggestion. We want to show that $\{B(t_2)-B(t_1),\dots,B(t_n)-B(t_{n-1})\}$ is an independent family of random variables where $t_1<\cdots<t_n$. Now that we know that this is Gaussian, we only need to show that $\mathbf{E}\,(B(t_{j+1})-B(t_j))(B(t_{k+1})-B(t_k)) = 0$ for $j\neq k$. But this is simple by assuming that $t_j<t_{j+1}\leq t_k<t_{k+1}$ which we can do by symmetry between j and k in the following computation, and so

$$\begin{aligned}\mathbf{E}\,(B(t_{j+1})-B(t_j))(B(t_{k+1})-B(t_k)) &= \mathbf{E}\,(B(t_{j+1})B(t_{k+1})) + \mathbf{E}\,(B(t_j)B(t_k)) \\ &\quad - \mathbf{E}\,(B(t_{j+1})B(t_k)) - \mathbf{E}\,(B(t_j)B(t_{k+1})) \\ &= t_{j+1}+t_j-t_{j+1}-t_j = 0.\end{aligned}$$

7. When $d=1$ and $X=B_t$, use integration by parts to show that

$$\mathbf{E}\,B_t^{2N} = t^N(2N-1)!! := t^N(2N-1)\times(2N-3)\times\dots3\times1$$

for every $N\geq 1$.
Suggestion. Denote $I_N=\mathbf{E}\,B_t^{2N}$. Then

$$I_{N+1} = c_t\int_{\mathbb{R}} x^{2N}x^2e^{-\frac{1}{2}x^2t^{-1}}\,\mathrm{d}x = -c_t t\int_{\mathbb{R}} x^{2N}x\,\partial_x(e^{-\frac{1}{2}x^2t^{-1}})\,\mathrm{d}x$$

Now since the exponential goes to zero faster than any polynomial we don't get any boundary terms from integration by parts and so

$$I_{N+1} = c_t\int_{\mathbb{R}} x^{2N}x^2e^{-\frac{1}{2}x^2t^{-1}}\,\mathrm{d}x = c_t t\int_{\mathbb{R}} e^{-\frac{1}{2}x^2t^{-1}}\partial_x x^{2N+1}\,\mathrm{d}x = (2N+1)tI_N$$

This gives a recursive equation for I_N, which can be solved by

$$I_N = I_1 \prod_{j=1}^{N-1} \frac{I_{j+1}}{I_j} = I_1 \prod_{j=1}^{N-1} t(2j+1) = I_1 t^{N-1}(2N-1)!!$$

But since $I_1 = \mathbf{E}\, B_t^2 = \min(t,t) = t$, the claim follows.

8. Using 7. show the Lemma 2.2 (iii) and (iv) from Lecture notes (page 14).
Suggestion. The (iii) is being done already, since we know that $B(t) - B(s) \sim B(t-s)$, and so $\mathbf{E}\,(B(t) - B(s))^2 = \mathbf{E}\, B(t-s)^2 = (t-s)$. Furthermore, the

$$\mathbf{E}\,(B(t) - B(s))^{2N} = \mathbf{E}\,(B(t-s)^{2N}) = (2N-1)!!(t-s)^N \leq \gamma_N |t-s|^N$$

by choosing $\gamma_N = (2N-1)!!$.

9. A π-system on a set S is a family $\mathscr{I} \neq \emptyset$ of subsets of S such that $\forall A, B \in \mathscr{I}$: $A \cap B \in \mathscr{I}$. Show that the set $\mathscr{J}_1 = \{\ (-\infty, x] :\ x \in \mathbb{R}\ \}$ is a π-system on $\mathbb{R}$.
Suggestion. Since $(-\infty, 0] \in \mathscr{J}_1$ we can be sure that $\mathscr{J}_1 \neq \emptyset$.

Let $A, B \in \mathscr{J}_1$. Then there are real numbers $x, y \in \mathbb{R}$Â such that $A = (-\infty, x]$ and $B = (-\infty, y]$. Since $A \cap B = (-\infty, \min(x,y)]$ and $\min(x,y) \in \mathbb{R}$, we have that $A \cap B \in \mathscr{J}_1$. Therefore, $\mathscr{J}_1$ is a π-system.

10. Let $f \colon \mathbb{R} \to \mathbb{R}_+$ be a simple, measurable function

$$f(x) = \sum_{k=1}^{n} a_k [\, x \in A_k\,].$$

Show that the time stationary Markov property for (X_t) with respect to $(\mathscr{F}_t)$ implies that

$$\mathbf{E}_x(f(X_t) \,|\, \mathscr{F}_s) = \mathbf{E}_{X_s}\, f\,(X_{t-s})$$

Suggestion. We have that

$$\mathbf{E}_x(f(X_t) \,|\, \mathscr{F}_s) = \sum_{k=1}^{n} a_k \mathbf{E}_x([\, X_t \in A_k\,] \,|\, \mathscr{F}_s) = \sum_{k=1}^{n} a_k \mathbf{P}_x\,(\, X_t \in A_k \,|\, \mathscr{F}_s\,)\,.$$

Therefore, by the time stationary Markov property

$$\mathbf{E}_x(f(X_t) \,|\, \mathscr{F}_s) = \sum_{k=1}^{n} a_k \mathbf{P}_{X_s}\,(\, X_{t-s} \in A_k\,) = \sum_{k=1}^{n} a_k \mathbf{E}_{X_s}\,[\, X_{t-s} \in A_k\,] = \mathbf{E}_{X_s}\, f\,(X_{t-s}).$$

11. Let $f\colon \mathbb{R} \to \mathbb{R}_+$ be a bounded, positive and measurable function (but not necessarily simple) and otherwise assume the same as in 10. Show that the claim of 10. holds in this case as well by using monotone convergence theorem.
Suggestion. Define for every $n > 0$ a simple function f_n such that

$$f_n(x) := \sum_{j=0}^{\infty} j2^{-n}[\, f(x) \in I_{j,n}\,]$$

We already know by Problem 4. that these converge monotonically to f. So, by monotone convergence

$$\mathbf{E}_x\left([\,A\,]f(X_t)\right) = \lim_{n\to\infty} \mathbf{E}_x\left([\,A\,]f_n(X_t)\right) = \lim_{n\to\infty} \mathbf{E}_x\left([\,A\,]\mathbf{E}_x(f_n(X_t)\,|\,\mathscr{F}_s)\,\right)$$

for every $A \in \mathscr{F}_s$. Now by the Problem 10. we can write

$$\lim_{n\to\infty} \mathbf{E}_x\left([\,A\,]\mathbf{E}_x(f_n(X_t)\,|\,\mathscr{F}_s)\,\right) = \lim_{n\to\infty} \mathbf{E}_x\left([\,A\,]\mathbf{E}_{X_s}\, f_n(X_{t-s})\,\right).$$

By using the monotone convergence theorem twice we get

$$\lim_{n\to\infty} \mathbf{E}_x\left([\,A\,]\mathbf{E}_{X_s}\, f_n(X_{t-s})\,\right) = \mathbf{E}_x\left([\,A\,]\lim_{n\to\infty} \mathbf{E}_{X_s}\, f_n(X_{t-s})\,\right) = \mathbf{E}_x\left([\,A\,]\mathbf{E}_{X_s}\, f(X_{t-s})\,\right).$$

Now since $\mathbf{E}_{X_s}\, f(X_{t-s})$ is $\mathscr{F}_s$-measurable, we get the claim. If the second usage of the monotone convergence theorem seems somewhat dangerous, don't worry, since

$$\mathbf{E}_x\left([\,A\,]\lim_{n\to\infty} \mathbf{E}_{X_s}\, f_n(X_{t-s})\,\right) = \int_A \lim_{n\to\infty} \mathbf{E}_{X_s(\omega)}\, f_n(X_{t-s})\,\mathbf{P}_x(\,\mathrm{d}\omega\,)$$

For every $\omega \in \Omega$ fixed, the integration inside is with respect to some $\mathbf{P}_z$ for some $z \in S$ so the monotone convergence theorem can be used pointwise for every ω.

The 12. is the original 10. This will be also be part of the next excercise sheet.

12. Let $(P_{t,x})$ be as in Lemma 3.13 in Lecture notes (page 22). Let

$$\mu^x_{(t_1,\ldots,t_n)}(A_1,\ldots,A_n) = \int_{A_1} P_{t_1,x}(\,\mathrm{d}x_1)\ldots\int_{A_n} P_{t_n-t_{n-1},x_{n-1}}(\,\mathrm{d}x_n)$$

Show that family of measures $\{\ \mu^x_{(t_1,\ldots,t_n,t_{n+1})} :\ x \in \mathbb{R}^d, 0 \le t_1 < \cdots < t_n\ \}$ satisfies the consistency condition for Kolmogorov Extension Theorem and deduce that therefore, there exists a stochastic process (X_t) such that $\mathbf{P}_x(X_t \in A) = \mu^x_t(A)$.