6. Stochastic integration

We have almost enough tools for the rest of the course but still we need the concept of stochastic integrals. This also provides a meaning for the stochastic differentiation.

6.1. **Stochastic integration with respect to Brownian motion, motivation.** Let's continue with the previous example of Q1 and Q2 in two dimensional case. We obtained the result by analysing the Brownian motion directly, since in 1-dimensional case the balls are intervals and the spheres are points. Trying to do the same in higher dimensional case would first correspond to attempt of using $|B_t|^2 - dt$, but then expectation of the stopping times appear so we don't get the similar equation for the probabilities.

However, if we notice that in 1-dimensional case $B_t = u(B_t)$ where u is a linear function and so it satisfies $u'' = 0$. Moreover, the other example of a martingale $B_t^2 - t = u(B_t, t)$ with $\partial_x^2 u = 2$ and $\partial_t u = -1$ and so it satisfies a parabolic equation $\partial_t u = \frac{1}{2}\partial_x^2 u$. We can try other polynomials that satisfy $\partial_t u = \frac{1}{2}\partial_x^2 u$ to see that $u(B_t, t)$ are martingales.

In 1-dimensional case only solutions of this equation $\partial_t u = \frac{1}{2}\triangle u$ which are constant in t are the affine functions. In higher dimensions there are much more solutions to the corresponding equation $\triangle u = 0$, but if we would know that $u(B_t)$ is a martingale whenever $\triangle u = 0$ we could approach the questions Q1 and Q2.

By looking at the Taylor formula in 1-dimensional case we notice that if u is smooth enough, we can write

$$\begin{aligned} u(B_t) - u(B_0) &= \sum_{k=1}^{n} u(B_{t_{k+1}}) - u(B_{t_k}) \\ &= \sum_{k=1}^{n} u'(B_{t_k})(B_{t_{k+1}} - B_{t_k}) + \sum_{k=1}^{n} \tfrac{1}{2} u''(B_{t_k})(B_{t_{k+1}} - B_{t_k})^2 + \dots \end{aligned}$$

where $0 = t_0 < t_1 < \cdots < t_{n+1} = t$ and where $\dots$ should vanish faster that the second order term. If we can find an increasing process A_t such that $A_{t_{k+1}} - A_{t_k} = (B_{t_{k+1}} - B_{t_k})^2$ we can express this as

$$\begin{aligned} u(B_t) - u(B_0) &= \sum_{k=1}^{n} u(B_{t_{k+1}}) - u(B_{t_k}) \\ &= \sum_{k=1}^{n} u'(B_{t_k})(B_{t_{k+1}} - B_{t_k}) + \tfrac{1}{2} \sum_{k=1}^{n} \int_{t_k}^{t_{k+1}} u''(B_{t_k})\, \mathrm{d}A_t + \dots \end{aligned}$$

Now if we make the discretisation denser and denser, we should have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int_{t_k}^{t_{k+1}} u''(B_{t_k})\, \mathrm{d}A_t + \cdots = \int_0^t u''(B_t)\, \mathrm{d}A_t$$

The first sum

$$M_n = \sum_{k=1}^{n} u'(B_{t_k})(B_{t_{k+1}} - B_{t_k})$$

is a discrete martingale, so we could ask for every $u \in C^2$ we can find an increasing adapted process A_t such that

$$u(B_t) - u(B_0) - \tfrac{1}{2}\int_0^t u''(B_s)\,\mathrm{d}A_s$$

is a martingale. This is the goal we want to achieve next.

6.2. **Quadratic variation.** In the sequel $(\mathscr{F}_t) = (\widetilde{\mathscr{F}_t})$ is a completed filtration. We will follow Revuz–Yor for the construction (there are few other options).

6.1. **Definition.** We denote $\mathscr{A}^+$ (resp. $\mathscr{A}$) the set of processes adapted to $(\mathscr{F}_t)$ that are have right continuous, finite and increasing (resp. of finite variation) paths almost surely.

We note that $A \in \mathscr{A}$ if and only if $\exists A^+, A^- \in \mathscr{A}^+$ such that $A = A^+ - A^-$. The process $A^+ + A^-$ is in $\mathscr{A}^+$ and it is called the *variation* of the process A.

6.2. **Preliminary definition.** Suppose K is progressively measurable and bounded on finite intervals almost surely. Then the *stochastic integral with respect to to* $A \in \mathscr{A}$ is defined

$$(X \cdot A)_t(\omega) := \int_0^t K_s\,\mathrm{d}A_s(\omega)$$

for ω such that K is bounded on finite intervals and $t \mapsto A_t(\omega)$ is of finite variation and $(X \cdot A)_t(\omega) = 0$ for other ω.

We have not yet introduced the progressively measurable processes, so let's do it now.

6.3. **Definition.** A process K is *progressively measurable*, if $(s,\omega) \mapsto K_s(\omega)$ as a mapping from $[0,t] \times \Omega \to \mathbb{R}$ is $\sigma(\mathscr{B}([0,t]) \times \mathscr{F}_t)$-measurable.

6.4. *Remark.* It might not be evident what kind of processes are progressively measurable, but every *right-continuous and adapted* process is progressively measurable and we will mostly be only dealing with such processes.

This does not yet cover martingales, since

6.5. **Lemma.** *A continuous martingale M that is in $\mathscr{A}$ is a constant.*

Proof. I'll add this soon. ∎

The following Theorem is the starting point of the stochastic integration. This provides the increasing adapted process we postulated in the motivation part.

6.6. **Definition.** A process X has a *finite quadratic variation*, if there exists a process $\langle X, X \rangle$ such that

$$\langle X, X \rangle_t = \lim \sum_k (X_{t_{k+1}\wedge t} - X_{t_k \wedge t})^2$$

as the $\sup_k |t_{k+1} - t_k| \to 0$ where the limit is a limit in probability[41].

6.7. **Theorem.** *A continuous and bounded martingale M has finite quadratic variation and $\langle M, M \rangle$ is the unique continuous process that belongs to $\mathscr{A}^+$ such that $\langle M, M \rangle_0 = 0$ and*

$$M^2 - \langle M, M \rangle$$

is a martingale.

Proof. Omitted. $\square$

This results does not yet cover Brownian motion, since it is not bounded martingale. We can extend this result by using stopping. As a first result let's prove the following.

6.8. **Lemma.** *For every stopping time τ we have*

$$\langle M^\tau, M^\tau \rangle = \langle M, M \rangle^\tau.$$

Proof. I'll add this soon. $\square$

With this and the definition of local martingales we can extend the Theorem 6.7 to continuous local martingales and so Brownian motion is covered as well.

6.9. **Theorem.** *Let M be a continuous local martingale. Then there exists a unique continuous process $\langle M, M \rangle \in \mathscr{A}^+$ such that $\langle M, M \rangle_0 = 0$ and*

$$M^2 - \langle M, M \rangle$$

is a continuous local martingale.

This can further extended by polarization.

[41] i.e. $X_n \to X$ in probability if $\forall \varepsilon > 0$: $\lim_n \mathbf{P}(|X_n - X| > \varepsilon) = 0$

6.10. **Theorem.** *Let M and N be a continuous local martingales. Then there exists a unique continuous process $\langle M, N \rangle \in \mathscr{A}$ such that $\langle M, N \rangle_0 = 0$ and*

$$MN - \langle M, N \rangle$$

is a continuous local martingale.

Proof. I'll add this soon. □

This allows us to define the *bracket* of N and M.

6.11. **Definition.** The process $\langle M, N \rangle$ is called the *bracket* of M and N.

6.12. **Example.** We already know the bracket of Brownian motion. Since $X_t = B_t^2 - t$ is a martingale, we have by uniqueness that $\langle B, B \rangle_t = t$.

6.13. **Lemma.** *For every stopping time τ we have*

$$\langle M^\tau, N^\tau \rangle \ = \langle M^\tau, N \rangle = \langle M, N^\tau \rangle = \langle M, N \rangle^\tau .$$

As a consequence of this we see that $\langle M, M \rangle$ does not vanish unless M is constant.

6.14. **Lemma.** *Suppose M is a continuous local martingale. We have that $\langle M, M \rangle = 0$ if and only if M is constant.*

Proof. I'll add this soon. □

6.3. **Stochastic integral with respect to a continuous semimartingale.** We will next define the stochastic integral in the sense of Itō with respect to a continuous semimartingale. But first, let's define the continuous semimartingales.

6.15. **Definition.** A process X is a *continuous semimartingale* with respect to the filtration $(\mathscr{F}_t)$, if $X = M + A$, where M is a continuous local martingale with respect to $(\mathscr{F}_t)$ and $A \in \mathscr{A}$ is such that $A_0 = 0$.

We have already defined the stochastic integral $(K \cdot X)$ for progressively measurable K and $X \in \mathscr{A}$. Following Revuz–Yor, let's define the class of integrals with respect to semimartingales, by assuming that we also know howto integrate with respect to continuous local martingales. Then in few steps we go backwards to smaller class of processes for which we can define the integration.

6.16. **Definition.** Suppose K is locally bounded process. Then we define

$$(K \cdot X)_t := (K \cdot M)_t + (K \cdot A)_t$$

where $X = M + A$ and M is a continuous local martingale. We will also denote

$$(K \cdot X)_t = \int_0^t K_s \, \mathrm{d}X_s$$

6.17. *Remark.* We notice that if $(K \cdot M)$ is a continuous local martingale, when M is a continuous local martingale, then $(K \cdot X)$ is a continuous semimartingale for every continuous semimartingale X and this follows from the theorem **??**.

We need to define the locally bounded process the make sense of the definition (6.16).

6.18. **Definition.** A locally bounded process K is progressively measurable process for which there exists an increasing sequence $(\tau_n) \uparrow \infty$ of stopping times and constants (C_n) such that

$$\forall n \colon |K^{\tau_n}| \leq C_n.$$

6.19. *Remark.* We notice that every bounded right-continuous adapted process is locally bounded. Furthermore, every continuous and adapted process is locally bounded.

In order to make the definition effective we need to define $(K \cdot M)$ for local martingales. This is the meaning of the follow result.

6.20. **Theorem.** *Suppose M is a continuous local martingale. For every $K \in L^2_{loc}(M)$ there exists a unique continuous local martingale $(K \cdot M)$ such that for every continuous local martingale N the following holds*

$$\langle \, K \cdot M, N \, \rangle = K \cdot \langle \, M, N \, \rangle.$$

Note that on the right-hand side $\langle \, M, N \, \rangle \in \mathscr{A}$ is of finite variation and so the stochastic integral on the right is defined at least if K is progressively measurable and as the following definition indicates that is the case. We will postpone the proof.

6.21. **Definition.** Suppose M is a continuous local martingale. We define

$$L^2_{\mathrm{loc}}(M) = \{ \, K : \; K \text{ is progressively measurable and } \exists (t_n) \uparrow \infty \colon \mathbf{E}_x \int_0^{\tau_n} K_s^2 \, \mathrm{d}\langle \, M, M \, \rangle_s < \infty . \}$$

6.22. *Remark.* This space of locally L^2-processes contain all locally bounded processes, since

$$\mathbf{E}_x \int_0^{\tau_n \wedge n} K_s^2 \, \mathrm{d}\langle \, M, M \, \rangle_s \leq C_n^2 \, \mathbf{E}_x \, \langle \, M, M \, \rangle_{\tau_n \wedge n} < \infty$$

as we will soon see.

In order to prove Theorem 6.20 we will first need the same result for continuous martingales that have bounded variance.

6.23. **Theorem.** *Suppose $M \in H^2$ is a continuous martingale. For every $K \in L^2(M)$ there exists a unique continuous martingale $(K \cdot M) \in H_0^2$ such that for every continuous martingale $N \in H^2$ the following holds*

$$\langle K \cdot M, N \rangle = K \cdot \langle M, N \rangle.$$

Moreover, the mapping $K \mapsto K \cdot M$ is an isometry[42] *between $L^2(M) \to H_0^2$.*

The idea of proving these theorems is simple. We use general result from functional analysis for Hilbert spaces together with Optional Stopping Theorem to deduce the existence and uniqueness of the martingale $(K \cdot M)$. We will postpone this proof for a while. But let's define the spaces H^2, H_0^2 and $L^2(M)$ so that we have a complete statement.

6.24. **Definition.** We define

$$H^2 = \{ M : \; M \text{ is a } L^2\text{-bounded continuous martingale} \}$$

and

$$H_0^2 = \{ M \in H^2 : \; M_0 = 0 \}$$

where M is L^2-*bounded* means that

$$\sup_t \mathbf{E} \, |M_t|^2 < \infty.$$

6.25. **Definition.** Suppose $M \in H^2$ is a continuous L^2-bounded martingale. We define

$$L^2(M) = \{ K : \; K \text{ is progressively measurable and} \\ \mathbf{E}_x \int_0^\infty K_s^2 \, \mathrm{d}\langle M, M \rangle_s < \infty. \}$$

Instead of proving the Theorem 6.23 let's show a special case of it to see what these different things mean.

6.26. **Example.** Let M be a continuous martingale. Let τ be the first exit time of M from the interval $(-N, N)$. Then $t \to M_{t \wedge \tau}$ is a in H^2, since

$$\mathbf{E} \, M_{t\wedge\tau}^2 \le \mathbf{E} \, M_\tau^2 < \infty$$

so we if we just denote $M = M^\tau$, we may assume that $M \in H^2$. The *elementary process* K which is defined as

$$K_t(\omega) = \sum_{j=1}^n [\, t_j \le s < t_{j+1} \,] K_{t_j}(\omega)$$

[42] i.e. it preserves the norm $\| K \|_{L^2(M)} = \| K \cdot M \|_{H^2}$

where K_{t_j} are $\mathscr{F}_{t_j}$-measurable and are uniformly bounded. Since K is right-continuous and adapted, it is progressively measurable. Moreover,

$$\mathbf{E}_x \int_0^\infty K_s^2 \,\mathrm{d}\langle\, M, M\,\rangle_s \;= \sum_{j=1}^n \mathbf{E}_x\, K_{t_j}^2 \int_{t_j}^{t_{j+1}} \,\mathrm{d}\langle\, M, M\,\rangle_s \;\le C\mathbf{E}_x \,\langle\, M, M\,\rangle_{\,t_{n+1}},$$

and since $M^2 - \langle\, M, M\,\rangle$ is a martingale, we have

$$\mathbf{E}_x \,\langle\, M, M\,\rangle_t \;= \mathbf{E}_x\, M_t^2 \;- x^2$$

and so, $K \in L^2(M)$. Let's verify that there exists a unique martingale $(K \cdot M)$. We make a guess that

$$(K \cdot M)_t = \sum_j K_{t_j}(M_{t_{j+1}\wedge t} - M_{t_j \wedge t})$$

or that the integral is given as a Riemann sum. This is martingale, since for every bounded stopping time ν we have

$$\mathbf{E}_x\,(K \cdot M)_\nu \;= \mathbf{E}_x \sum_j K_{t_j}(M^\nu_{t_{j+1}} - M^\nu_{t_j}) \;= \mathbf{E}_x \sum_j K_{t_j}\mathbf{E}\,\left(M^\nu_{t_{j+1}} - M^\nu_{t_j} \mid \mathscr{F}_{t_j}\right)$$

where the second identity follows from the $\mathscr{F}_{t_j}$-measurability of K_{t_j}. Since M^ν is a martingale, the right-hand side is 0. But since $(K \cdot M)_0 = 0$ by definition, we have shown that $\mathbf{E}_x\,(K \cdot M)_\nu \;= \mathbf{E}_x\,(K \cdot M)_0$ which is equivalent with the martingale property.

We still need to verify that $(K \cdot M)$ satisfies the

$$\langle\, K \cdot M, N\,\rangle = K \cdot \langle\, M, N\,\rangle$$

for every $N \in H^2$. The right-hand side is

$$(K \cdot \langle\, M, N\,\rangle)_t = \sum_j K_{t_j}(\langle\, M, N\,\rangle_{t\wedge t_{j+1}} - \langle\, M, N\,\rangle_{t \wedge t_j}).$$

Therefore, we should show that

$$(K \cdot M)N - \sum_j K_{t_j}(\langle\, M, N\,\rangle^{t_{j+1}} - \langle\, M, N\,\rangle^{t_j})$$

is a martingale to verify the claim and since by our guess

$$(K \cdot M)N = \sum_j K_{t_j}(M^{t_{j+1}}N - M^{t_j}N)$$

this reduces to

$$\sum_j \mathbf{E}_x\, K_{t_j}\mathbf{E}\,\left(M_\nu^{t_{j+1}}N_\nu - M_\nu^{t_j}N_\nu - \langle\, M, N\,\rangle_\nu^{t_{j+1}} + \langle\, M, N\,\rangle_\nu^{t_j} \mid \mathscr{F}_{t_j}\right) \;= 0$$

for every bounded stopping time ν. But this follows from the identity $\langle\, M, N\,\rangle^\eta = \langle\, M^\eta, N\,\rangle$, since for instance

$$M^{t_j}N - \langle\, M, N\,\rangle^{t_j} = M^{t_j}N - \langle\, M^{t_j}, N\,\rangle$$

is a martingale by the definition of bracket.

6.4. **Itō formula.** Next we will introduce the Itō formula.

6.27. **Theorem** (Itō formula)**.** *Let $X = (X^j)$ be a continuous d-dimensional semimartingale and $f \in C^2(\mathbb{R}^d, \mathbb{R})$. Then $f(X_t)$ is a continuous semimartingale and moreover,*

$$f(X_t) - f(X_0) = \sum_{j=1}^{d} \int_0^t \partial_j f(X_t)\, \mathrm{d}X_t^j + \tfrac{1}{2} \sum_{j,k=1}^{d} \int_0^t \partial_{jk} f(X_t)\, \mathrm{d}\langle X^j, X^k \rangle_t$$

almost surely.

Here the d-dimensional semimartingale (resp. local martingale) means a process $X = (X^1, \ldots, X^d)$ such that each coordinate process X^j is a semimartingale (resp. local martingale). In the same way we could talk about complex d-dimensional semimartingales and local martingales, just by requiring the real and imaginary parts to be d-dimensional semimartingales and local martingales, respectively.

We notice already one thing. If some of the components of X, say X^j, is of finite variation, then $\langle X^j, X^k \rangle = \langle X^k, X^j \rangle = 0$. This make the assumption $\partial_{jk} f$ is continuous seem superfluous since we are integrating with respect to zero measure. Just by making tiny changes the following stronger result can be shown.

6.28. **Theorem** (Itō formula (Version II))**.** *Let $X = (X^j)$ be a continuous d-dimensional semimartingale and let's assume that $X^{l+1}, \ldots, X^d$ are of finite variation. If $f \in C^1(\mathbb{R}^d, \mathbb{R})$ and $(f^1, \ldots, f^l) \in C^2(\mathbb{R}^l, \mathbb{R})$, then $f(X_t)$ is a continuous semimartingale and*

$$f(X_t) - f(X_0) = \sum_{j} \int_0^t \partial_j f(X_t)\, \mathrm{d}X_t^j + \tfrac{1}{2} \sum_{j,k \le l} \int_0^t \partial_{jk} f(X_t)\, \mathrm{d}\langle X^j, X^k \rangle_t$$

almost surely. In differential form[43] *the formula becomes*

$$\mathrm{d}f(X_t) = \sum_{j} \partial_j f(X_t)\, \mathrm{d}X_t^j + \tfrac{1}{2} \sum_{j,k \le l} \partial_{jk} f(X_t)\, \mathrm{d}\langle X^j, X^k \rangle_t$$

The Itō formula follows from the Weierstraß Theorem for approximating continuous functions on compact sets with polynomials. The result for polynomials follow from the Integration by Parts formula.

[43] by differential form we mean that insted of writing $\int_0^t \mathrm{d}X_s$ we write just $\mathrm{d}X_t$

6.29. **Theorem** (Integration by Parts)**.** *If X and Y are two continuous semimartingales, then*

$$X_tY_t = X_0Y_0 + \int_0^t X_s \, \mathrm{d}Y_s + \int_0^t Y_s \, \mathrm{d}X_s + \langle X, Y \rangle_t$$

almost surely or in differential form

$$\mathrm{d}(XY)_t = X_t \, \mathrm{d}Y_t + Y_t \, \mathrm{d}X_t + \mathrm{d}\langle X, Y \rangle_t$$

Proof. I'll add this soon as well as some approximation results. □

6.5. **Stochastic dominated convergence theorem and other results.** Here we collect some results we used in proving the Itō formula. These are used later on as well.

6.30. **Theorem** (Dominated convergence for stochastic integral)**.** *Let X be a continuous semimartingale. Suppose (K^n) is a sequence of locally bounded processes and K is locally bounded process such that $|K^n| \leq K$ for every n, and $K_t^n \to 0$ for every t almost surely, then*

$$\sup_{0 \leq t \leq \rho} \int_0^t K_s^n \, \mathrm{d}X_s \to 0$$

in probability for every ρ.