9. Tanaka's formula and local time

We have a reasonable comprehensive picture on the stopped Brownian motion that is stopped before leaving a given domain $G \subset \mathbb{R}^d$. We will next study the reflecting Brownian motion (let's call it R) in a given domain G.

The process R should behave exactly like a Brownian motion inside the domain and when it "hits" the boundary, it should reflect towards the normal direction. This already means that the domain cannot be arbitrarily rough, since we must be able to say what the normal direction is.

The simplest possible geometry for G in $\mathbb{R}^d$ is the case of a half-space $G = \mathbb{R}^{d-1} \times (0, \infty)$ with a *flat* boundary $\partial G = \mathbb{R}^{d-1} \times \{0\}$.

Intuitively we see that since the reflection is towards the last coordinate, then we should have $R = (B^1, \dots, B^{d-1}, R^d)$, where $(B^1, \dots, B^{d-1}, B^d)$ is a d-dimensional Brownian motion. Since outside the half plance, namely when $R^d = 0$, the processes should be independent, we should have $R^d = f(B^d)$. This brings us the 1-dimensional case.

In the 1-dimensional case, it seems evident that $R_t = |B_t|$ is the reflecting Brownian motion. We already know that R is a continuous submartingale and by Doob–Meyer decomposition, we know that it is a continuous semimartingale, i.e. we can write $R = M + A$, with some continuous local martingale M and continuous, increasing and adapted process A.

Since our ultimate goal is to generalize this to higher dimensional case as well with more complicated domains G, we should find out what M and A are.

Unfortunately, $f(x) = |x|$ is not a C^2-function, so we cannot just use Itō formula to find out the decomposition. But if we would formally compute the derivatives (or compute the weak derivatives), we obtain that $f'(x) = [\, x > 0\,] - [\, x < 0\,]$ for almost every x. Now the second derivative becomes $2\delta_0$ so it has mass only at 0 with total mass 2.

Formally this would suggest that

$$R_t = R_0 + \int_0^t ([\, B_s > 0\,] - [\, B_s < 0\,])\, \mathrm{d}B_s + \int_0^t \delta_0(B_s)\, \mathrm{d}s.$$

It turns out that we are really on the right track, since we can generalize Itō to handle this case, since the absolute value function f is convex. For this we introduce a left handed derivative, f'_- which is just

$$f'_-(x) = \lim_{h \downarrow 0} \frac{f(x-h) - f(x)}{-h}$$

9.1. **Example.** (1) When $f \in C^1$, then $f' = f'_-$. Moreover, the left handed derivative is linear operation.

(2) When $f(x) = x^+$, then f is continuosly differentiable outside 0, so $f'_-(x) = f'(x) = [\, x > 0\,]$ whenever $x \neq 0$. When $x = 0$, then

$$f'_-(0) = \lim_{h\downarrow 0} \frac{0-0}{-h} = 0$$

so $f'_-(x) = [\, x > 0\,]$ for every x.

(3) When $f(x) = x^-$, then f is continuosly differentiable outside 0, so $f'_-(x) = f'(x) = -[\, x < 0\,]$ whenever $x \neq 0$. When $x = 0$, then

$$f'_-(0) = \lim_{h\downarrow 0} \frac{h-0}{-h} = -1$$

so $f'_-(x) = -[\, x \leq 0\,]$ for every x.

(4) When $f(x) = |x| = x^+ + x^-$, we have by the linearity that $f'_-(x) = [\, x > 0\,] - [\, x \leq 0\,]$ for every x.

9.2. **Theorem.** *Suppose X is a continuous semimartingale and $f\colon \mathbb{R} \to \mathbb{R}$ a convex function. Then there exists a continuous increasing process A^f such that*

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)\,\mathrm{d}X_s + \tfrac{1}{2}A^f_t$$

Proof. Suppose first that f is convex and C^2. Then $f'_- = f'$ and we can use Itō formula to deduce that

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)\,\mathrm{d}X_s + \tfrac{1}{2}\int_0^t f''(X_s)\,\mathrm{d}\langle\, X, X\,\rangle_s$$

Since f is convex, its second derivative f'' is positive, i.e. $f''(X_s) \geq 0$. Thus, if we denote

$$A^f_t := \int_0^t f''(X_s)\,\mathrm{d}\langle\, X, X\,\rangle_s$$

we have found the A^f is continuous and increasing and it satisfies the conditions we had for it.

When f is just convex (like the absolute value function), then we still know that it is Lipschitz continuous on closed and bounded intervals. Moreover, it has a left handed derivative at every point. To proceed we approximate the function f by a sequence of smooth convex functions f_n which we define as

$$f_n(x) := n\int_{\mathbb{R}} f(x+y)\psi(ny)\,\mathrm{d}y = n\int_{\mathbb{R}} f(y)\psi(n(y-x))\,\mathrm{d}y$$

where $\psi \geq 0$ is a C^∞-function that is 0 outside an interval $[a, b]$ where $a < b < 0$. We also require that the integral of ψ is one, i.e.

$$\int_{\mathbb{R}} \psi(x)\,\mathrm{d}x = 1.$$

Since f is Lipschitz continuous on the interval $[x+a/n, x+b/n]$, it is bounded there and thus, f_n is well defined and continuous function for every n. Moreover, this implies that we may use dominated convergence to take the differentiation inside the integral and thus,

$$f_n'(x) = -n^2 \int_{\mathbb{R}} f(y)\psi'(n(y-x))\,\mathrm{d}y$$

and we obtained that $f_n \in C^1$. We can repeat this differentiation as many times as we please, so $f_n \in C^\infty$. Furthermore, since by change of variable $ny = z$, we find that

$$f_n(x) = \int_{\mathbb{R}} f(x+\frac{z}{n})\psi(z)\,\mathrm{d}z$$

so the dominated convergence theorem implies that

$$\lim_{n\to\infty} f_n(x) = \int_{\mathbb{R}} \lim_{n\to\infty} f(x+\frac{z}{n})\psi(z)\,\mathrm{d}z = f(x)\int_{\mathbb{R}} \psi(z)\,\mathrm{d}z = f(x).$$

We also find out (Excercise) that f_n is also a convex function and $f_n'(x) \to f_-'(x)$ for every x. We can even find out[46] that $|f_-'(x)| \geq |f_n'(x)|$.

The first part of the proof showed that there are continuous and increasing processes A^{f_n} such that

$$f_n(X_t) = f_n(X_0) + \int_0^t f_n'(X_s)\,\mathrm{d}X_s + \tfrac{1}{2}A_t^{f_n}$$

Since $f_n(x) \to f(x)$ for every x, we deduce that

$$f(X_t) = f(X_0) + \lim_{n\to\infty}\Big(\int_0^t f_n'(X_s)\,\mathrm{d}X_s + \tfrac{1}{2}A_t^{f_n}\Big)$$

We want to show that the stochastic integrals converge as well and for this, we will use the dominated convergence for stochastic integrals. If we denote $K_t^n = f_n'(X_t)$, then $K_t^n \to f_-'(X_t)$ and $|K_t^n| \leq |f_-'(X_t)|$. Since f is Lipschitz on a closed and bounded intervals, f_-' is bounded. Therefore, we can use the dominated convergence for stochastic integrals (Theorem 6.30) and we find out that

$$\lim_{n\to\infty}\int_0^t f_n'(X_s)\,\mathrm{d}X_s = \int_0^t f_-'(X_s)\,\mathrm{d}X_s$$

in probability uniformly for every $t \leq \rho$. We can therefore choose a subsequence (f_{n_k}) so that this convergence is uniform almost surely. This implies that

$$\tfrac{1}{2}\lim_{n_k\to\infty} A_t^{f_{n_k}} = f(X_t) - f(X_0) - \int_0^t f_-'(X_s)\,\mathrm{d}X_s$$

[46] this will be added to the suggested solutions of the excercises

uniformly for every $t \leq \rho$. If we now just define $A^f_t := \lim_{n_k\to\infty} A^f_{n_k}$ we are done, since A^f is continuous as a uniform limit of continuous processes almost surely and increasing as a limit of increasing processes. □

This result gives us the Tanaka formula, which provides the semimartingale decomposition of the reflecting Brownian motion $R = |B|$.

9.3. **Theorem** (Tanaka's formula)**.** *For every $a \in \mathbb{R}$, there exists an increasing continuous process L^a such that*

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a)\,\mathrm{d}X_s + L^a_t$$

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t [\, X_s > a\,]\,\mathrm{d}X_s + \tfrac{1}{2}L^a_t$$

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t [\, X_s \leq a\,]\,\mathrm{d}X_s + \tfrac{1}{2}L^a_t$$

where $\operatorname{sgn}(x) = [\, x > 0\,] - [\, x \leq 0\,]$.

9.4. **Definition.** The process L^a is called *the local time of the semimartingale X at the point a.*

Proof. Since $f_1(x) = (x-a)^+$ and $f_2(x) = (x-a)^-$ are both convex functions, we can use the Theorem 9.2 and we obtain that

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t [\, X_s > a\,]\,\mathrm{d}X_s + \tfrac{1}{2}A^{f_1}_t$$

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t [\, X_s \leq a\,]\,\mathrm{d}X_s + \tfrac{1}{2}A^{f_2}_t$$

for some continuous and increasing processes A^{f_1} and A^{f_2} which might be different. However, by subtracting these identities and using the fact that $f_1(x) - f_2(x) = (x-a)^+ - (x-a)^- = x - a$ we obtain

$$\begin{aligned}(X_t - a) &= (X_0 - a) + \int_0^t ([\, X_s > a\,] + [\, X_s \leq a\,])\,\mathrm{d}X_s + \tfrac{1}{2}(A^{f_1}_t - A^{f_2}_t) \\ &= (X_0 - a) + X_t - X_0 + \tfrac{1}{2}(A^{f_1}_t - A^{f_2}_t) = X_t - a + \tfrac{1}{2}(A^{f_1}_t - A^{f_2}_t)\end{aligned}$$

and therefore, $A^{f_1} = A^{f_2}$ so we can define $L^a_t = A^{f_1}_t$ so that the last two identities are now shown. Since $|x - a| = (x-a)^+ + (x-a)^-$ and $[\, x > a\,] - [\, x \leq a\,] = \operatorname{sgn}(x - a)$ we obtain by summing the last two identities together that

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a)\,\mathrm{d}X_s + \tfrac{1}{2}(L^f_t + L^f_t)$$

and the claim is shown. □

Returning to the reflecting Brownian motion $R_t = |B_t|$ we can now express it in terms of a local martingale and an increasing process, namely

$$R_t = R_0 + \int_0^t \operatorname{sgn}(B_s)\,\mathrm{d}B_s + L_t$$

where $L_t = L_t^0$ is the local time of the Brownian motion at 0. We are nearly done with the properties of 1-dimensional reflecting Brownian motion, since next we show that $\mathrm{d}L_t = 0$ whenever $B_t \neq 0$ or that L only increases when $B_t = 0$. Once we have shown that we see that

$$R_t^\tau - R_{\tau_0}^\tau = \int_{\tau_0}^{\tau\wedge t} \operatorname{sgn}(B_s)\,\mathrm{d}B_s$$

for every stopping times $\tau_0 < \tau$ such that $B_s \neq 0$ for every $\tau_0 \leq s \leq \tau$.

If we denote

$$\beta_t := \int_0^t \operatorname{sgn}(B_s)\,\mathrm{d}B_s$$

which is a continuous local martingale with respect to the $(\mathscr{F}_t^B)$, the previous identity may be written as

$$R_t^\tau - R_{\tau_0}^\tau = \beta_t^\tau - \beta_{\tau_0}^\tau.$$

The process $\beta$is really Brownian motion by the Lévy's Characterization Theorem. In the following, we say that a process X is a $(\mathscr{F}_t)$-Brownian motion, if it has the same law as Brownian motion and the $X_t - X_s$ is independent of the σ-algebra $\mathscr{F}_s$ for every $t > s$.

9.5. **Theorem** (Lévy's Characterization Theorem)**.** *Suppose X is a $(\mathscr{F}_t)$-adapted continuous d-dimensional process and suppose $X_0 = 0$. Then the following are equivalent*

1. *X is a $(\mathscr{F}_t)$-Brownian motion.*
2. *$X = (X^1, \ldots, X^d)$ is a d-dimensional continuous local $(\mathscr{F}_t)$-martingale and $\langle X^j, X^k \rangle_t = [\, j = k\,]t$.*
3. *$X = (X^1, \ldots, X^d)$ is a d-dimensional continuous local $(\mathscr{F}_t)$-martingale and for every for every $f = (f_1, \ldots, f_d)$ with $f_j \in L^2(\mathbb{R}^+)$ the process*

$$Y_t^f := \exp\Big(i \sum_{k=1}^d \int_0^t f_k(X_s)\,\mathrm{d}X_s^k + \tfrac{1}{2} \sum_{k=1}^d \int_0^t f_k^2(X_s)\,\mathrm{d}s\Big)$$

is a complex and bounded martingale.

Proof. This is left to excercises. □

We can now verify that β is Brownian motion. First of all it is a continuous local martingale with respect to the $(\mathscr{F}_t^B)$, the completed filtration generated by the Brownian motion B. Moreover, its quadratic variation process is

$$\langle \beta, \beta \rangle_t = \int_0^t (\operatorname{sgn}(B_s))^2 \, \mathrm{d}\langle B, B \rangle_s = \int_0^t 1 \, \mathrm{d}s = t$$

so β is a $(\mathscr{F}_t^B)$-Brownian motion by the Lévy's Characterization Theorem.

We have almost shown that the $R = |B|$ satisfies the requirements we should have for a reflecting Brownian motion in 1-dimensional case when it reflects from 0. We still need to verify that the local time does not increase when $B_t \neq 0$. This follows from the next result.

9.6. **Theorem.** *Suppose X is a continuous semimartingale. For every $t \geq 0$ we have*

$$\int_0^t |X_s - a| \, \mathrm{d}L_s^a = 0$$

almost surely.

Proof. We will use both Itō formula for the semimartingales $Y_t = |X_t - a|$ and $Z_t = X_t - a$ with a function $f(x) = x^2$. Since $f(Y_t) = |X_t - a|^2 = (X_t - a)^2 = f(Z_t)$, we can then compare the two different representations. First of all

$$f(Z_t) - f(Z_0) = 2\int_0^t Z_s \, \mathrm{d}Z_s + \langle Z, Z \rangle_t = 2\int_0^t (X_s - a) \, \mathrm{d}X_s + \langle X, X \rangle_t$$

Secondly,

$$f(Y_t) - f(Y_0) = 2\int_0^t Y_s \, \mathrm{d}Y_s + \langle Y, Y \rangle_t = 2\int_0^t |X_s - a| \, \mathrm{d}Y_s + \langle X, X \rangle_t$$

Therefore, everything else being the same we have

$$\int_0^t (X_s - a) \, \mathrm{d}X_s = \int_0^t |X_s - a| \ \mathrm{d}Y_s$$

By the Tanaka formula, we know that $\mathrm{d}Y_s = \operatorname{sgn}(X_s - a) \, \mathrm{d}X_s + \mathrm{d}L_s^a$ and so

$$\int_0^t |X_s - a| \ \mathrm{d}Y_s = \int_0^t |X_s - a| \operatorname{sgn}(X_s - a) \ \mathrm{d}X_s + \int_0^t |X_s - a| \, \mathrm{d}L_s^a$$

Since $|x| \operatorname{sgn}(x) = x$ for every x, we obtain that

$$\int_0^t (X_s - a) \, \mathrm{d}X_s = \int_0^t |X_s - a| \ \mathrm{d}Y_s = \int_0^t (X_s - a) \, \mathrm{d}X_s + \int_0^t |X_s - a| \, \mathrm{d}L_s^a$$

which implies the claim. □