8. Stochastic differential equations

We would like to generalise the Kakutani's representation theorem from Brownian motion and harmonic functions to more general partial differential equations.

Therefore, let's denote by L the following second order differential operator

$$L\colon f \mapsto \Big(x \mapsto \tfrac{1}{2}\sum_{j,k} A^{jk}(x) f_{jk}(x) + \sum_{j} b^j(x) f_j(x)\Big).$$

This generalises the Laplace operator, since if we choose $A^{jk}(x) = [\, j = k \,]$ and $b^j(x) = 0$, we obtain that $L = \frac{1}{2}\triangle$.

Suppose we have a function $u\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ that is once continuosly differentiable in the first coordinate, say t, and twice continuosly differentiable with respect to the rest of the coordinates, say x.

Then we can use Itō formula to compute, when

$$Z_t = u(s-t, X_t)$$

is a local martingale on $[0, s)$. We would like to link this problem (when Z is a local martingale) with the differential operator L. As an axample, if X is the Brownian motion or $L = \frac{1}{2}\triangle$, then we notice that

$$\mathrm{d}Z_t = \mathrm{d}M_t - \partial_t u(s-t, B_t)\,\mathrm{d}t + \tfrac{1}{2}\triangle u(s-t, B_t)\,\mathrm{d}t.$$

This Z is a local martingale if $\partial_t = \frac{1}{2}\triangle u$ for $(0, s) \times \mathbb{R}^d$, which gives the link between the Laplace operator and Brownian motion we are after.

We can generalize this by making the ansatz that X is satisfies a following stochastic differential equation

$$\mathrm{d}X_t := \sigma(X_t)\,\mathrm{d}B_t + c(X_t)\,\mathrm{d}t, \quad X_0 = x \tag{8.1}$$

for some *matrix valued* function $\sigma\colon z \mapsto (\sigma^{ij}(z))$ and for a *vector valued* function $c\colon \mathbb{R}^d \to \mathbb{R}^d$.

If we apply Itō formula to the process $Z_t = u(s-t, X_t)$ then as we with the Brownian motion, we obtain that

$$\mathrm{d}Z_t = -\partial_t u(s-t, X_t)\,\mathrm{d}t + \sum_j u_j(s-t, X_t)\,\mathrm{d}X_t^j + \tfrac{1}{2}\sum_{j,k} u_{jk}(s-t, X_t)\,\mathrm{d}\langle X^j, X^k \rangle_t.$$

If we assume that $Lu = 0$, as with Brownian motion and if in addition $\mathrm{d}\langle X^j, X^k \rangle_t = A^{jk}(X_t)\,\mathrm{d}t$ and if $c(X_t) = b(X_t)$, then

$$\begin{aligned} \mathrm{d}Z_t &= \Big(-\partial_t u(s-t, X_t) + \sum_j u_j(s-t, X_t) b^j(X_t) \\ &\qquad + \tfrac{1}{2} \sum_{j,k} A^{jk}(X_t) u_{jk}(s-t, X_t) \Big)\,\mathrm{d}t + \mathrm{d}M_t \\ &= Lu(s-t, X_t)\,\mathrm{d}t + \mathrm{d}M_t = \mathrm{d}M_t \end{aligned}$$

so we obtain a same link between Z being a local martingale and u solving the partial differential equation $Lu = 0$. We have already found c, since we just really assumed $c = b$. The matrix σ needs to be derived, but we can find it by computing $\mathrm{d}\langle X^j, X^k \rangle_t$. Since $X_t = N_t + A_t$, the bracket $\langle X^j, X^k \rangle_t = \langle N^j, N^k \rangle_t$, where

$$\mathrm{d}N_t^j = \sum_k \sigma^{jk}(X_t)\,\mathrm{d}B_t^j .$$

This implies that

$$\mathrm{d}\langle X^j, X^k \rangle_t = \sum_{l,m} \sigma^{jm}(X_t) \sigma^{kl}(X_t)\,\mathrm{d}\langle B^m, B^l \rangle_t .$$

Since we already know that $\mathrm{d}\langle B^m, B^l \rangle_t = [\, m = l \,]\,\mathrm{d}t$, we obtain

$$\mathrm{d}\langle X^j, X^k \rangle_t = \sum_l \sigma^{jl}(X_t)(\sigma^\top)^{lk}(X_t)\,\mathrm{d}t = (\sigma\sigma^\top)^{jk}(X_t)\,\mathrm{d}t,$$

where the matrix function $\sigma^\top$ is the *transpose* of the matrix functin σ. So we require that σ satisfies $\sigma\sigma^\top = A$,

How can we utilise this nowobtained martingale property? By using Optional Stopping.

8.2. **Lemma.** *Suppose that X satisfies the stochastic differential equation* (8.1)*. Suppose further, that u is satisfies a initial value problem*

$$\begin{cases} \partial_t u = Lu, & \textit{joukossa } (0,\infty) \times \mathbb{R}^d \\ u(0,x) = f(x) & \textit{jokaisella } x \in \mathbb{R}^d \end{cases} \tag{8.3}$$

and moreover, u is bounded and twice continuosly differentiable with respect to x and once continuosly differentiable with respect to t If the coefficients in the equation (8.1) *are chosen that $c = b$ and $\sigma\sigma^\top = A$, then we have a representation*

$$u(x,t) = \mathbf{E}_x\, f(X_t)\,.$$

Proof. We already deduced above that $Z_t = u(s-t, X_t)$ is a continuous local martingale in the interval $[0,s)$. Since u is bounded, the process Z is uniformly integrable and therefore, we have

$$Z_s = \lim_{t\uparrow s} Z_t = u(0, X_s) = f(X_s).$$

Since $\mathbf{E}_x Z_s = \mathbf{E}_x Z_0 = u(s,x)$, the claim follows. □

This shows that there is a certain duality between the existence of the solution to the parabolic initial value problem (8.3) between the uniqueness of the stochastic differential equation (8.1), since if the assumptions of the previous lemma is satisfied, then

$$u(t,x) = \mathbf{E}_x f(X_t) = \mathbf{E}_x f(\widetilde{X}_t)$$

for every x, for every t and for every f whenever X and $\widetilde{X}$ are two (possibly different) solutions to the stochastic differential equation (8.1) starting at x. This however implies that X_t and $\widetilde{X}_t$ are identically distributed given they start at the same point x.

Similarly, if we assume the existance solution to the stochastic differential equation (8.1), the previous lemma shows the uniqueness of the classical solutions to the initial value problem.

8.4. *Remark.* We could also ask whether there is a duality between the the uniqueness of the solutions of the initial value problem (8.3) and the existence of the solutions to the stochastic differential equations. We notice that the uniqueness should imply that the existence of the transition functions (P_t) of a Markov process and for those we do have an existence theorem. Verifying that however takes some effort.

The other dual question would be if the uniqueness of the stochastic differential equation (8.1) would could ask if this implies the solvability of the initial value problem (8.3).

These are connected via socalled *martingale problem* which we, however, have to leave outside of the course. Uniqueness of the solutions to the stochastic differential equation would imply that the solutions are strong Markov processes and this like in the case of Kakutani's representation theorem leads towards showing the existence since we used Markov property in showing that the $w(x) = \mathbf{E}_x f(B_\tau)$ was a solution to the boundary value problem.

8.1. **Itō's existence and uniqueness result for SDEs.** We have noticed that solving partial differential equations and stochastic differential equations are closely connected. Next we will show that under certain assumptions we can solve the stochastic differential equation in the same way as an ordinary differential equation. The following is the classical result by Itō (1942).

Since integration is the inverse operation to the differentiation, the stochastic differential equation really is the stochastic integral equation

$$X_t = X_0 + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s + \int_0^t b(s, X_s)\,\mathrm{d}s \tag{8.5}$$

where $b\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}_+ \times \mathbb{R}^r \to \mathbb{R}^d$ are Borel measurable functions.

8.6. **Theorem.** *Suppose that b and σ satisfy the local Lipschitz condition*

$$\forall n \exists K \forall |x|, |y| \le n\colon |b(t,x) - b(t,y)| + \|\, \sigma(t,x) - \sigma(t,y) \,\| \le K|x-y|.$$

Then the solutions of the stochastic integral equation (8.5) *are pathwise unique.*

If we assume more on the coefficients we can prove the existence as well.

8.7. **Theorem.** *Suppose that b and σ satisfy the global Lipschitz condition on the time interval* $[0, s]$

$$\forall |x|, |y|\colon |b(t,x) - b(t,y)| + \|\, \sigma(t,x) - \sigma(t,y) \,\| \le K|x-y|.$$

together with the condition of linear growth

$$|b(t,x)| + |\sigma(t,x)| \le C(1 + |x|)$$

If X_0 is independent from the given Brownian motion B and it has a finite second moment, then the equation (8.5) *has a unique strong solution.*

Before we can prove these we need to *define* what is meant by stroung solutions and pathwise uniqueness.

8.8. **Definition.** Suppose we are given $(\Omega, \mathscr{F}, \mathbf{P})$ and a filtration $(\mathscr{F}_t)$ on Ω. We call the pair (X, B) a $(\mathscr{F}_t)$-weak *solution* to the stochastic differential equation (8.5) if

i) X is continuous d-dimensional stochastic processes adapted to $(\mathscr{F}_t)$

ii) B is continuous r-dimensional Brownian motion with respect to $(\mathscr{F}_t)$

iii) we have

$$\int_0^t \|\, \sigma(s, X_s) \,\|^2 + |b(s, X_s)|\,\mathrm{d}s < \infty$$

for $\mathbf{P}_x$-almost surely for every x

iv) together, the processes X and B satisfy the equation (8.5).

Now that we have defined a (weak) solution, we can define the strong solution.

8.9. **Definition.** Suppose we are given $(\Omega, \mathscr{F}, \mathbf{P})$ and a filtration $(\mathscr{F}_t)$ on Ω. We call the process X a strong solution to the stochastic differential(8.5), if (X, B) is an $(\widetilde{\mathscr{F}_t^B})$-weak solution.

8.10. *Remark.* It is clear that if the stochastic differential equation (8.5) has a strong solution then it always has a weak solution, but the other direction does not hold in general. We'll postpone the example for a while.

The uniqueness has also two different flavours.

8.11. **Definition.** Suppose (X, B) and (X', B) are $(\mathscr{F}_t)$-weak solutions and $(\mathscr{F}'_t)$-weak solutions respectively with the same Brownian motion. Suppose $X_0 = X'_0$ almost surely. We say that the solution is *pathwise unique*, if this implies that $\forall t : X_t = X'_t$ holds $\mathbf{P}$-almost surely.

8.12. **Definition.** Suppose (X, B) and (X', B') are $(\mathscr{F}_t)$-weak solutions and $(\mathscr{F}'_t)$-weak solutions possibly on different probability spaces $(\Omega, \mathscr{F}, \mathbf{P})$ and $(\Omega', \mathscr{F}', \mathbf{P}')$ respectively. Suppose $X_0 = X'_0$ in distribution, i.e. we suppose that $\forall U\colon \mathbf{P}(X_0 \in U) = \mathbf{P}'(X'_0 \in U)$. We say that the solution is *unique in law*, if this implies that $\mathbf{P}\left(\forall t \in F : X_t \in U_t\right) = \mathbf{P}'(\forall t \in F : X'_t \in U_t)$ for every $F = \{t_1, \ldots, t_n\}$.

8.13. *Remark.* This time it is no longer so clear that pathwise uniqueness implies uniqueness in law, since the probability spaces can change. This, however, does hold by the result of Yamada and Watanabe (1971). The other direction does not hold and the example will, again, be postponed.

Actually, Yamada and Watanabe showed that the existence of a weak solution together with pathwise uniqueness implies that the solutions are strong solutions.