7. Recurrence, transience and Kakutani

7.1. **Recurrence and transience.** Let's return to the questions that partially settled the claim that that the Brownian motion lives in the whole space. This was done by analysing the following questions:

Q1. Let $x_0, x \in \mathbb{R}^d$. Will Brownian motion always hit the point x when $B_0 = x_0$?

Q2. Let $x_0, x \in \mathbb{R}^d$ and $r > 0$. Will Brownian motion always hit the the ball of radius r and centered at the point x when $B_0 = x_0$?

Q3. Let $x_0 \in \mathbb{R}^d$, and $r > 0$. Will Brownian motion always hit the complement of the ball of radius r and centered at the origin?

which we pharaphrased with the help of first hitting time as:

Q1. Let $x_0, x \in \mathbb{R}^d$. Is $\mathbf{P}_{x_0}\left(\tau_{\{x\}} < \infty\right) = 1$?

Q2. Let $x_0, x \in \mathbb{R}^d$ and $r > 0$. Is $\mathbf{P}_{x_0}\left(\tau_{B_r(x)} < \infty\right) = 1$?

Q3. Let $x_0 \in \mathbb{R}^d$ and $r > 0$. Is $\mathbf{P}_{x_0}\left(\tau_{B_r(0)^C} < \infty\right) = 1$?

We already showed that Q3 has an affirmative answer in any dimensions. We showed that Q1 and Q2 also hold when $d = 1$. But we were lacking a map that would translate the problem from higher dimensions to a question on an interval. And for this we can use the Itō formula since this was the one motivation for the introduction of the stochastic integration to begin with.

Let's use the terminalogy that if the hitting probability to a set is 1, then the Brownian motion is *recurrent* to the set. If not, then we say that the Brownian motion is *transient* to the set.

If a set happens to such that the probability of hitting it is 0 then we call the set *polar*.

In 1-dimensional case, we used Brownian motion itself to the first exit time τ from the interval (a, b). Since Brownian motion is a martingale, we deduced that for $x \in (a, b)$ we have

$$\mathbf{P}_x\left(B_\tau = a\right) = \frac{b - x}{b - a}, \quad \mathbf{P}_x\left(B_\tau = b\right) = \frac{x - a}{b - a}$$

In higher dimensional case, we can change the coordinates so that the center of the ball in Q2 is origin and in Q1 the point we are hitting is the origin. This makes the setting invariant with respect to rotations and since Brownian motion is invariant under rotations, we can turn the problem to 1-dimensional by considering the modulus (or the squared modulus) of the Brownian motion, i.e.

$$X_t = |B_t|^2 = \sum_j (B_t^j)^2$$

Now if we set $\tau_r := \inf\{\, t > 0 : \ X_t = r \,\}$, then the questions become *i*) is $\mathbf{P}_x\,(\,\tau_0 < \infty\,) = 1\,$?

ii) is $\mathbf{P}_x\,(\,\tau_r < \infty\,) = 1$ when $r > 0\,$?

Since we are in 1-dimensional setting already, let's study these simultaneously by looking at the following problem: what is the probability

$$p(x, r, R) := \mathbf{P}_x\,(\,\tau_r < \tau_R\,) =?$$

when $r < |x| < R$. We can use the Itō formula to answer this. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}$ be (hopefully) C^2-function and let $Z_t = \varphi(X_t)$. *If Z_t is a continuous local martingale* and $\tau = \inf\{\ t > 0 :\ \ X_t \notin (r, R)\ \}$, then by Optional Stopping Theorem

$$\mathbf{E}_x\, Z_\tau^{\tau_n} \ = \mathbf{E}_x\, Z_0 \ = \varphi(x)$$

where $\tau_n \uparrow \infty$ and Z^{τ_n} is UI martingale. If φ is continuous over the closed interval $[r, R]$, then we can take the limit $n \to \infty$ and we obtain that

$$\mathbf{E}_x\, Z_\tau \ = \varphi(x).$$

But since

$$\mathbf{E}_x\, Z_\tau \ = \varphi(r)\mathbf{P}_x\,(\,X_\tau = r\,) + \varphi(R)\mathbf{P}_x\,(\,X_\tau = R\,) = \varphi(R) + p(x, r, R)(\varphi(r) - \varphi(R)).$$

we can deduce that

$$\mathbf{P}_x\,(\,\tau_r < \tau_R\,) = p(x, r, R) = \frac{\varphi(R) - \varphi(x)}{\varphi(R) - \varphi(x)} \tag{7.1}$$

So computing the probability is reduced in finding a function φ that turns the process Z into a continuous local martingale. And what would be better than the Itō formula, since

$$\mathrm{d}Z_t = \varphi'(X_t)\,\mathrm{d}X_t + \tfrac{1}{2}\varphi''(X_t)\,\mathrm{d}\langle\, X\,\rangle_t.$$

Let's compute what is $\mathrm{d}X_t$, so that we can find the local martingale and the finite variation part. Since $X_t = f(B_t)$ where $f(x) = x_1^2 + \cdots + x_d^2$, we have

$$\mathrm{d}X_t = \sum_j 2B_t^j\,\mathrm{d}B_t^j + \sum_j\ \mathrm{d}t = \ \mathrm{d}M_t + d\,\mathrm{d}t.$$

Since the quadratic variation of a semimartingale is the quadratic variation of its martingale part we have $\langle\, X\,\rangle_t = \langle\, M\,\rangle_t$ and therefore,

$$\mathrm{d}\langle\, X\,\rangle_t = \sum_{j,k} 4B_t^k B_t^j\,\mathrm{d}\langle\, B^k, B^j\,\rangle_t = \sum_j 4(B_t^j)^2\,\mathrm{d}t = 4X_t\,\mathrm{d}t$$

where we used the fact that the components of Brownian motion are independent which implies that $\langle\, B^k, B^j\,\rangle = 0$ (Ex). For this computation, we used the formula

$$\langle\, H \cdot M, K\ \cdot N\,\rangle = HK \cdot \langle\, M, N\,\rangle$$

This implies that

$$\mathrm{d}Z_t = \mathrm{d}N_t + n\varphi'(X_t)\,\mathrm{d}t + 2\varphi''(X_t)X_t\,\mathrm{d}t.$$

and thus, Z_t is a continuous local martingale if

$$d\varphi'(x) + 2\varphi''(x)x = 0 \Longrightarrow \frac{f'(x)}{f(x)} = -\frac{d}{2x} \Longrightarrow f(x) = Cx^{-d/2}$$

where $f(x) := \varphi'(x)$. If we choose $C = 1$, then by integrating f we find the fucntion φ, which therefore is of form

$$\varphi(x) = \begin{cases} \ln|x|, & d = 2, \\ C_d x^{(2-d)/2}, & d \geq 3, \end{cases}$$

Let's assume first that $d = 2$. Then

$$\mathbf{P}_x\left(\tau_r < \infty\right) \geq \mathbf{P}_x\left(\tau_r < \tau_R\right) = \frac{\ln R - \ln x}{\ln R - \ln r} \to 1$$

when $R \uparrow \infty$, so Q2 has an affirmative answer in plane or Brownian motion is recurrent to discs irregardless how small it is and how far it is. But Q1 has a different answer in plane.

Since

$$\mathbf{P}_x\left(\tau_0 < \tau_R\right) \leq \mathbf{P}_x\left(\tau_r < \tau_R\right) = \frac{\ln R - \ln x}{\ln R - \ln r} \to 0$$

when $r \downarrow 0$, we have $\mathbf{P}_x\left(\tau_0 < \tau_R\right) = 0$ for every $x \neq 0$ and for every $R > |x|$. Since Brownian motion is continuous, $\tau_R \uparrow \infty$, as $R \uparrow \infty$. Therefore,

$$\mathbf{P}_x\left(\tau_0 < \infty\right) = 0$$

for every $x \neq 0$. In the Excercise sheet 4 we extend this by strong Markov property to cover even the case $x = 0$ and so $\mathbf{P}_x\left(\tau_0 < \infty\right) = 0$ for every $x \in \mathbb{R}^2$. This means that single points are *polar sets* in plane or *Brownian motion will (almost surely) never hit a given point in a plane irregardless where it starts.*

The state of affairs change more for $d \geq 3$. Then $\alpha := \frac{2-d}{2} < 0$ and by monotone convergence

$$\mathbf{P}_x\left(\tau_r < \infty\right) = \lim_{R\to\infty} \mathbf{P}_x\left(\tau_r < \tau_R\right) = \lim_{R\to\infty} \frac{R^\alpha - x^\alpha}{R^\alpha - r^\alpha} = \frac{x^\alpha}{r^\alpha} < 1$$

as long as $x > r$, so Brownian motion will not hit a ball of radius $r > 0$ with positive probability *irregardless* how large the ball is or how close we started. So Brownian motion is *transient* from balls for $d \geq 3$. When we still shrink $r \downarrow 0$, we find out that the single points are polar sets also in dimensions $d \geq 3$.

7.2. **Dirichlet boundary value problem and Kakutani's representation theorem.** Let's return to the connection between harmonic functions and stopped (and as we see soon killed) Brownian motion.

We want to show compute $u(x) = \mathbf{E}_x\, f(B_\tau)$, when $G \subset \mathbb{R}^d$ is an open, connected bounded domain and $x \in \overline{G}$ is a given point, the stopping time τ is the first exit time from the domain G and $f\colon \Gamma \to \mathbb{R}$ is a given continuous function on the boundary of the domain $\Gamma = \partial G$. The Kakutani's representation theorem says that this computation is equivalent with solving the Dirichlet problem for the Laplace equation, i.e. under some conditions u should be a *harmonic function* inside the domain G, which is continuous upto the boundary and would coinicde with the function f there.

Let's show first that having a harmonic function w in the classical sense in G is equivalent with $Z_t := w(B_t)$ being a local martingale on the interval $[0,\tau)$.

The other direction $\triangle w = 0$ in G implies that Z is a local martingale for every starting points $x \in G$ follows from the Itō formula. First of all, when $x \in G$ the distance from the boundary is strictly positive and therefore $\mathbf{P}_x\,(\,\tau > 0\,) = 1$ by the continuity of Brownian motion. This means that Z_t is well defined on an interval $[0,\tau)$.

If we assume that $w \in C^2(G)$ and is continuous upto the boundary, we can apply Itō formula gives that Z is continuous semimartingale on $[0,\tau)$ since Brownian motion is and

$$\mathrm{d}Z_t = \sum_j w_j(B_t)\,\mathrm{d}B_t^j + \tfrac{1}{2}\sum_{j,k} w_{jk}(B_t)\,\mathrm{d}\langle\, B^j, B^k\,\rangle_t$$

Since the coordinates of Brownian motion are independent, we have that $\langle\, B^j, B^k\,\rangle_t = t[\,j = k\,]$ (Ex.). All in all, we obtain

$$\mathrm{d}Z_t = \sum_j w_j(B_t)\,\mathrm{d}B_t^j + \tfrac{1}{2}\sum_j w_{jj}(B_t)\,\mathrm{d}t =:\ \mathrm{d}M_t + \tfrac{1}{2}\triangle w(B_t)\,\mathrm{d}t =:\ \mathrm{d}M_t + \mathrm{d}A_t.$$

The first term is a continuous local martingale on a random interval $[0,\tau)$ and the latter is of finite variation.

7.2. **Lemma.** *Suppose $w \in C^2(G)$. Process Z is a continuous local martingale on the interval $[0,\tau)$ for every starting points $x \in G$ if and only if $\triangle w(x) = 0$ for every $x \in G$.*

Proof. The Itō formula implies immediately that Z is a local martingale on $[0,\tau)$ for every starting point x if $\triangle w = 0$ in G.

Let as now *assume* that Z is a continuous local martingale for every starting point $x \in G$. By the above identity we have that in this case $A_t = Z_t - M_t$ is a continuous local martingale. However, it is also of finite variation, but this implies that A is constant and since $A_0 = 0$ almost surely, we have $A \equiv 0$ almost surely. This in turn implies that $A_\eta = 0$ almost surely for every stopping time $\eta < \tau$.

We will show $A_\eta = 0$ for every stopping time $\eta < \tau$ implies that $\triangle w = 0$. Suppose for contrary that $\triangle w(x) > 0$. Since $g \in C^2(G)$, we have that $\triangle w$ is continuous and hence we may found a positive numbers $r > 0$ and $s > 0$ such that $\triangle w(y) \geq r$ for every $y \in B_s(x) \subset G$. If we take η is the first exit time from $B_s(x)$, then the continuity of Brownian motion implies that $\eta > 0$ and $\eta < \tau$. Therefore,

$$A_\eta = \int_0^\eta \triangle w(B_s)\,\mathrm{d}s \geq \eta r$$

when we start from the point $x \in U$. This mean that assuming $\triangle w \neq 0$ implies that $A_\eta \neq 0$ for some stopping time $\eta < \tau$, and hence if Z is a continuous local martingale, we have that $\triangle w = 0$. $\square$

With this lemma and Optional Stopping Theorem we can show the representation part of the Kakutani's representation theorem, namely that every *classical* harmonic function with a given Dirichlet value is given by a functional of Brownian motion.

7.3. **Lemma.** *If a boundary value problem*

$$\begin{cases} \triangle w = 0 & \textit{alueessa } G \\ w = f & \textit{reunalla } \Gamma \end{cases}$$

admits a solution $w \in C^2(G) \cap C(\overline{G})$, then $w(x) = \mathbf{E}_x\, f(B_\tau)$ for every $x \in G$.

Proof. If $\triangle w = 0$ in G and $w \in C^2(G)$, then the previous lemma implies that Z is a continuous local martingale on the interval $[0, \tau)$. Therefore, we can find a sequence (τ_n) of stopping times so that $\tau_n \uparrow \tau$ and that Z^{τ_n} is a uniformly integrable martingale. This means that by Optional Stopping Theorem

$$w(x) = \mathbf{E}_x\, Z_0 = \mathbf{E}_x\, Z_0^{\tau_n} = \mathbf{E}_x\, Z^{\tau_n}(\tau) = \mathbf{E}_x\, w(B(\tau_n)$$

for every $n \in \mathbb{N}$. Since $w \in C(\overline{G})$ and $B_{\tau_n} \in \overline{G}$, we know that $|w(B(\tau_n))| \leq \|\, w \,\|_\infty$ and we can use the Lebesgue dominated convergence theorem to deduce that

$$w(x) = \mathbf{E}_x \lim_{n\to\infty} w(B(\tau_n)\,.$$

Since Brownian motion and w are continuous functions and $\tau_n \uparrow \tau < \infty$ we may deduce that

$$w(x) = \mathbf{E}_x\, w(B_\tau)$$

Since $w = f$ and $B_\tau \in \Gamma = \partial G$, we may deduce that

$$w(x) = \mathbf{E}_x\, f(B_\tau)$$

which proves the claim. $\square$

7.4. *Remark.* We notice that if this representation holds also an a point $x \in \Gamma$, then

$$w(x) = \mathbf{E}_x\, f(B_\tau)\; = f(x).$$

As we will soon see, most of the points on the boundary have a property that the Brownian motion leaves the domain G immediately, i.e.

$$\mathbf{P}_x\,(\,\tau = 0\,) = 1.$$

We will call points that satisfy $\mathbf{P}_x\,(\,\tau = 0\,) \;=\; 1$ *regular boundary points.* So we will soon see that most of the points are regular boundary points and the representation formula extends to regular boundary points, since for every such a point x, we have

$$w(x) = \mathbf{E}_x\, f(B_\tau)\; = f(x).$$

We will next show that the if $w(x) := \mathbf{E}_x\, f(B_\tau)$ and all of the boundary points are regular, then w is harmonic function, $w \in C^2(G) \cap C(\overline{G})$ and $w = f$ on Γ.

We first show that $w \in C^2(G)$. Then we verify that $X_t := w(B_t)$ is a continuous local martingale on $[0, \tau)$ for every starting point $x \in G$. Then the Lemma 7.2 already shows that $\triangle w = 0$ in G. As a last step, we show that w can be continuously extended to the regular points and as we noted it will then coincide with the boundary data, i.e. $w(x) = f(x)$ for every regular boundary points.

7.5. **Lemma.** *The function $w(x) := \mathbf{E}_x\, f(B_\tau)$ belongs to $C^\infty(G)$.*

Proof. Suppose $x \in G$ and $r > 0$ are such that $D_r(x) \subset G$. Let's denote by η_r the first exit time from the ball $D_r(x)$ with radius r and center x.

Since B has the strong Markov property, we have

$$w(x) = \mathbf{E}_x\, f(B_\tau)\; = \mathbf{E}_x\, \mathbf{E}\;\left(f(B_\tau)\,|\,\mathscr{F}_\eta\right)\; = \mathbf{E}_x\, \mathbf{E}_{B(\eta)}\, f(B_\tau)\; = \mathbf{E}_x\, w(B_\eta)\,.$$

(the exact details will be left to excercises). Since f is continuous function on a compact set Γ, it is bounded and therefore, $\|\, w\,\|_\infty \le \|\, f\,\|_\infty < \infty$ i.e. w is bounded as well. From this we can deduce (Ex) that u is actually infinitely differentiable at the point $x \in G$, which means that $u \in C^\infty(G)$. $\square$

Let's next deduce the local martingale property for $Z_t = u(B_t)$.

7.6. **Lemma.** *The process* $Z_t = u(B_t)$ *is a continuous local martingale on* $[0, \tau)$ *for every starting point* $x \in G$.

Proof. Since we know that $u \in C^\infty(G)$ and Brownian motion is continuous, Z is continuous on interval $[0, \tau)$. Let's denote

$$Y_t = \mathbf{E}_x(f(B_\tau) \,|\, \mathscr{F}_t) \ .$$

Since B is a strong Markov process, we have that

$$[\, s < \tau \,]\mathbf{E}_x(f(B_\tau) \,|\, \mathscr{F}_s) \ = [\, s < \tau \,]\mathbf{E}_{B(s)}\, f(B_\tau) \ = [\, s < \tau \,]u(B_s).$$

or in other words, $[\, s < \tau \,]Y_s = [\, s < \tau \,]Z_s$. This implies that for every $\eta < \tau$ the stopped processes Y^η and Z^η coincide. Since Brownian motion is a continuous local martingale on $[0, \tau)$ we can choose a sequence $\tau_n \uparrow \tau$ so that $\tau_n < \tau$ for every n. Since $f(B_\tau)$ is integrable, Y is a uniformly integrable martingale and therfore, $Z^{\tau_n} = Y^{\tau_n}$ is uniformly integrable martingale for every n. This, however, means that Z is a continuous local martingale on $[0, \tau)$. □

7.7. **Corollary.** *The function* $w(x) := \mathbf{E}_x\, f(B_\tau)$ *is harmonic in* G.

Proof. This follows from Lemma 7.2 since $w \in C^2(G)$ by Lemma 7.5 and Z is a continuous local martingale for every starting point $x \in G$ by Lemma 7.6. □

We will still need to verify that u is continuous upto the boundary. For this we need to *assume* that the boundary ∂G is regular, i.e. every point $x \in \partial G$ is a regular point.

7.8. **Lemma.** *Suppose* $x \in \partial G$ *is a regular point, i.e.* $\mathbf{P}_x\,(\, \tau = 0 \,) = 1$. *Then* $w(x) = f(x)$ *and* u *is continuous at* x *in the sense that* $w(x_n) \to w(x)$ *for every* $(x_n) \subset G$*, such that* $x_n \to x$.

Proof. The fact that $w(x) = f(x)$ for regular points was already mentioned in the Remark 7.4. Let's verify the continuity along a given sequence. So let $x \in \Gamma = \partial G$ be a regular point and $(x_n) \subset G$ be a sequence that convergence to x, i.e. $x_n \to x$. Since for every $z \in \overline{G}$ and every $\delta > 0$ it holds that

$$u(z) = \mathbf{E}_z\, f(B_\tau) \ = \mathbf{E}_z\, f(B_\tau)[\, B_\tau \in D(x, \delta) \,] + \mathbf{E}_z\, f(B_\tau)[\, B_\tau \notin D(x, \delta) \,]$$

we can deduce by the continuity of the function f that

$$u(z) = f(x)\mathbf{P}_z\,(\, B_\tau \in D(x, \delta) \,) + \mathbf{E}_z\, f(B_\tau)[\, B_\tau \notin D(x, \delta) \,] + \mathrm{o}(1),$$

where $\mathrm{o}(1)$ vanishes as $\delta \to 0$ uniformly with respect to $z \in G$. So given $\varepsilon > 0$, we can choose $\delta > 0$ so small that

$$\sup_n \left| w(x_n) - f(x)\mathbf{P}_{x_n}\,(\, B_\tau \in D(x, \delta) \,) - \mathbf{E}_{x_n}\, f(B_\tau)[\, B_\tau \notin D(x, \delta) \,] \right| < \varepsilon$$

If $\mathbf{P}_{x_n}(\, B_\tau \in D(x,\delta)\,) \to 1$, as $x_n \to x$, then we obtain that

$$\limsup_{n\to\infty} |w(x_n) - f(x)| \ < \varepsilon$$

which implies the continuity since $f(x) = w(x)$ for the regular point x and since $\varepsilon > 0$ is arbitrary. This limit estimate follows since the boundedness of the domain G implies that every continuous function on the boundary is bounded and hence

$$|\mathbf{E}_{x_n} f(B_\tau)[\, B_\tau \notin D(x,\delta)\,]\,| \le \|\, f\,\|_\infty \mathbf{P}_{x_n}(\, B_\tau \notin D(x,\delta)\,) \to 0.$$

as $n \to \infty$ and thus

$$|w(x_n) - f(x)| \ \le \varepsilon + 2\|\, f\,\|_\infty \mathbf{P}_{x_n}(\, B_\tau \notin D(x,\delta)\,) \to \varepsilon$$

To finish the proof, we should show that for every $\delta > 0$ it holds that $\mathbf{P}_{x_n}(\, B_\tau \in D(x,\delta)\,) \to 1$, as $n \to \infty$. This claim is quite convincing since it says that closer from the boundary point x we start, the first exit place should be close to point x as well.

This in turn follows from the "intuitively" evident claim that for every $t > 0$ the probability $\mathbf{P}_{x_n}(\, \tau > t\,) \to 0$, as $n \to \infty$. This claim in a way says that closer we start from the boundary, less time it should take to exit and in the limit it should not take any time, which seems like the definition of the regular point. Proving this claim is left to Excercises.

So suppose we know that for every $t > 0$, we have that $\mathbf{P}_{x_n}(\, \tau > t\,) \to 0$, as $n \to \infty$. Then we can rewrite

$$\begin{aligned}
\mathbf{P}_{x_n}(\, B_\tau \in D(x,\delta)\,) &= \mathbf{P}_{x_n}(\, \tau > t, B_\tau \in D(x,\delta)\,) + \mathbf{P}_{x_n}(\, \tau \le t, B_\tau \in D(x,\delta)\,) \\
&= \mathbf{P}_{x_n}(\, \tau \le t\,) + \mathbf{P}_{x_n}(\, \tau > t, B_\tau \in D(x,\delta)\,) - \mathbf{P}_{x_n}(\, \tau \le t, B_\tau \notin D(x,\delta)\,)\,.
\end{aligned}$$

for every $t > 0$. This means that together with the assumed fact that the probability $\mathbf{P}_{x_n}(\, \tau > t\,) \to 0$, as $n \to \infty$ the claim

$$\lim_{n\to\infty} \mathbf{P}_{x_n}(\, B_\tau \in D(x,\delta)\,) = 1$$

is equivalent with the claim that for some $t > 0$ the probabilities

$$\mathbf{P}_{x_n}(\, \tau \le t, B_\tau \notin D(x,\delta)\,)$$

are arbitrarily small for every large enough large n. When n is large enough, x_n is at most the distance of $\delta/2$ from the point x. If we now "draw" a ball of radius $\delta/2$ around the point x_n it will be (by the triangle inequality) be inside the ball $D(x,\delta)$.

This means that

$$\mathbf{P}_{x_n}(\, \tau \le t, B_\tau \notin D(x,\delta)\,) \le \mathbf{P}_{x_n}(\, \eta_n \le t\,)$$

where η_n is the first exit time of the Brownian motion from the ball $D(x_n, \delta/2)$, since in order to leave from the domaing G outside the ball $D(x, \delta)$ for the first time, it must have left the ball $D(x_n, \delta/2)$ before that.

Changing the coordinates (so that x_n becomes the origin, we see that

$$\mathbf{P}_{x_n}(\,\eta_n \leq t\,) = \mathbf{P}_0(\,\eta \leq t\,)$$

where η is the first exit time of the Brownian motion from the ball $D(0, \delta/2)$. This, however, can made arbitrarily small by choosing $t > 0$ small enough since Brownian motion is continuous. To see this, we notice that for every $t_n \downarrow 0$, the events $A_n = \{\eta \leq t_n\}$ is a monotonically decreasing and hence

$$\mathbf{P}_0(\,\eta = 0\,) \ = \lim_{n\to\infty} \ \mathbf{P}_0(\,\eta \leq t_n\,)\,.$$

Since Brownian motion is continuous, the probability on the left is zero. So, let's choose $t > 0$ so small that $\mathbf{P}_0(\,\eta \leq t\,) \ < \varepsilon$ which implies that

$$\mathbf{P}_{x_n}(\,\tau \leq t, B_\tau \notin D(x, \delta)\,) \leq \mathbf{P}_{x_n}(\,\eta_n \leq t\,) = \mathbf{P}_0(\,\eta \leq t\,) < \varepsilon$$

holds for every large enough n. This finishes the proof. □

We can now combine all these to the following theorem.

7.9. **Theorem.** *Suppose every $x \in \Gamma$ is regular. Then for every continuous function f on the boundary Γ, the Dirichlet problem has a unique solution $u(x) = \mathbf{E}_x\, f(B_\tau)$ that is C^2 inside G and continuous upto the boundary.*

Proof. Everything else is clear, so we only need to verify that u is continuous in $\overline{G}$. Suppose we have a sequence $(x_n) \subset \overline{G}$ that converges to $x \in \Gamma$. If $x_n \in G$ for infinitely many n, then we see that $(u(x_n))$ has a convergencing subsequence that converges to $u(x)$. Otherwise, all but finitely many $x_n \in \Gamma$. Let $\varepsilon > 0$ and choose a point $z_n \in G$ such that $|x_n - z_n| < 1/n$ and $|u(x_n) - u(z_n)| < \varepsilon$. By construction $z_n \to x$ and

$$|u(x_n) - u(x)| \leq |u(x) - u(z_n)| + \varepsilon$$

The previous result shows now that $u(z_n) \ \to u(x)$ and so

$$\limsup_{n\to\infty} |u(x_n) - u(x)| \leq \varepsilon$$

which means that also $u(x_n) \to u(x)$. □

What kind of points are regular ? We can easily deduce that if a part of the boundary is flat (i.e. part of some hyperplane), then at least these points are regular, since 0 is a regular point for Brownian motion on the interval $[0, 1]$ (Ex). If a point $x \in \Gamma$ is not regular, then $\mathbf{P}_x(\,\tau = 0\,) < 1$. However, the event $\{\tau = 0\} \in \mathscr{F}_{0+}$ and this σ-algebra is quite trivial.

7.10. **Theorem** (Blumenthal's 0-1 law)**.** *If* $A \in \mathscr{F}_{0+}$*, then either* $\mathbf{P}\,(\,A\,) = 0$ *or* $\mathbf{P}\,(\,A\,) = 1$*.*

Therefore, a point which is not regular satisfies $\mathbf{P}_x\,(\,\tau = 0\,) = 0$. This suggests that an irregular point should be almost surrounded by interior points.

7.11. **Example.** If $G = D(\bar{0}, 1) \setminus \{\bar{0}\} \subset \mathbb{R}^2$ is a punctured disc then the point $\bar{0}$ is not regular. This is since Brownian motion does not return the point and it is continuous so it cannot "travel" to the outer boundary in zero time.

7.12. **Definition.** A point $x \in \Gamma$ satisfies a *flat cone condition*, if there is a $d-1$-dimensional hyperplane $L \ni x$, a $d-1$-dimensional ball $U \ni x$ in the hyperplane L and a $d-1$-dimensional cone $V \subset L$ with strictly positive aperture[44], with the tip at point x such that $U \cap V \subset G^C$.

7.13. **Example.** If one can find a cone (i.e. d-dimensional cone), that after truncation by a ball is in G^C, then x satisfies the cone condition, which is therefore strictly stronger than the flat cone condition.

Moreover, if the boundary is differentiable at $x \in \Gamma$, then x satisfies (flat) cone condition. More generally, if the the boundary can be represented as a graph of a Lipschitz function in some neighborhood of x, then it also satisfies a (flat) cone condition.

7.14. **Example.** In two dimensional case (i.e. in the plane) the domain $G = D(\bar{0}, 1) \setminus \gamma$ which is the unit disc with a proper[45] line segment γ removed, then every point in $\partial G = \partial D(\bar{0}, 1) \cup \gamma$ satisfies the flat cone condition.

7.15. *Remark.* Actually, in the two dimensional case and if G is open (like we are assuming all the time), a stronger regularity result holds, namely, if a point $x \in \partial G$ can be connected with an arc $\gamma \subset G^C$ to another point $y \in G^C$, then x is regular.

7.16. **Example.** In dimensions three or higher, the inwardd cusps are too

7.17. **Lemma.** *If* $x \in \partial G$ *satisfies the flat cone condition, then* x *is a regular point.*

Proof. Since Brownian motion is invariant under rigid motions (i.e. rotations and dilations), we may assume that $x = \bar{0}$ is the origin and the flat cone $C \subset \{x_d = 0\}$.

[44] so it contais interior points

[45] i.e. the segment is not a single point

Define a sequence of stopping times

$$\tau_n = \inf\{\, t > 1/n : \ B_t^d = 0 \,\}$$

which are the times the Brownian motion after some delay $1/n$ hit the hyperplane supporting the flat cone C for the first time. By construction, the sequence (τ_n) is decreasing and since in one dimensional case the boundary of an interval $[0,1]$ is regular, we have also that $\tau_n \downarrow 0$ for $\mathbf{P}_0$-almost surely.

If we denote by $X = (B^1, \ldots, B^{d-1})$, then $(X_{\tau_n}, 0) \in G^C$ whenever $X_{\tau_n} \in C_0$, where $C_0 = C \cap U$ is the truncated flat cone. Therfore, if $X_{\tau_n} \ \in C_0$, we also have that $\tau \le \tau_n$ and so

$$\mathbf{P}_0\,(\,\tau \le \tau_n\,) \ge \mathbf{P}_0\,(\,X_{\tau_n} \ \in C_0\,)$$

Moreover, since $\tau_n \downarrow 0$, we have by monotone convergence that

$$\mathbf{P}_0\,(\,\tau = 0\,) \ge \limsup_{n\to\infty} \mathbf{P}_0\,(\,X_{\tau_n} \ \in C_0\,)$$

Therefore, if we can show that the right-hand side is strictly positive, the regularity follows from the Blumenthal 0-1 -law.

This on the other hand follows, if we can show that

$$\mathbf{P}_0\,(\,X_{\tau_n} \in C\,) = \theta > 0$$

for every n, since then

$$\mathbf{P}_0\,(\,X_{\tau_n} \ \in C_0\,) = \mathbf{P}_0\,(\,X_{\tau_n} \ \in C, X_{\tau_n} \in U\,) = \theta - \mathbf{P}_0\,(\,X_{\tau_n} \ \in C, X_{\tau_n} \notin U\,)$$

and the probability $\mathbf{P}_0\,(\,X_{\tau_n} \notin U\,)$ goes to zero as $n \to \infty$, since X is continuous and $\tau_n \downarrow 0$ for $\mathbf{P}_0$-almost surely.

This last part follows from the independence of B^d and X the fact that C is a cone. First by independence,

$$\mathbf{P}_0\,(\,X_{\tau_n} \in C\,) = \mathbf{E}_0\, g\,(\tau_n)$$

where

$$g(t) = \mathbf{P}_0\,(\,X_t \in C\,)\,.$$

Now since $X_t \sim t^{1/2} X_1$ and moreover, since C is a cone, $t^{1/2} C = C$, we notice that

$$g(t) = \mathbf{P}_0\left(\,X_1 \in t^{1/2} C\,\right) = \mathbf{P}_0\,(\,X_1 \in C\,) = g(1)$$

so g is a constant function, implying that

$$\mathbf{P}_0\,(\,X_{\tau_n} \in C\,) = \mathbf{E}_0\, g\,(\tau_n) = g(1) = \mathbf{P}_0\,(\,X_1 \in C\,)$$

The last probability is strictly positive, since the cone has a positive Lebesgue measure and X_1 is a Gaussian random variable i.e. it shares the null sets with Lebesgue measure. $\square$