5. More on martingales and local martingales

Recall that if (X_t) is an $(\mathscr{F}_t)$-supermartingale, then $\mathbf{E}\, X_t \le \mathbf{E}\, X_s$ for $t > s$ and if (X_t) is a martingale, then $\mathbf{E}\, X_t = \mathbf{E}\, X_s$ for every $t > s$. This naturally does not yet imply martingale (nor supermartingale) property.

5.1. **Example.** Let (X_t) be a continuous, $(\mathscr{F}_t)$-adapted and integrable stochastic process with $\mathbf{E}\, X_t = \mathbf{E}\, X_0$ for every t. This does not mean that it will be a martingale, since $X_t = B_t^3$ satisfies all these conditions with respect to history of the Brownian motion. Since $x^3 - y^3 = (x-y)^3 + 3y(x-y)^2 + 3y^2(x-y)$, we have for every $t > s$ that

$$\mathbf{E}\,\left(B_t^3 - B_s^3 \,|\, \mathscr{F}_s\right) = 0 + 3B_s \mathbf{E}\, B_{t-s}^2 + 0 = 3(t-s)B_s.$$

Since the right-hand side does not vanish, we don't have a martingale at hand. Furthermore, it is neither sub- or supermartingale, since B_s is not always positive or negative.

However, now that we know about stopping times, we can verify a following quite remarkable observation which says that if in the previous example we would have assumed that $\mathbf{E}\, X_\tau = \mathbf{E}\, X_0$ for every *bounded stopping time* τ, the process X would have been a martingale.

5.2. **Lemma.** *Let (X_t) be (right continuous) adapted with respect to $(\mathscr{F}_t)$. Suppose for every $s \ge 0$ and for every bounded proper stopping time $\tau \ge s$ we have that X_τ is integrable and furthermore, we have that $\mathbf{E}\, X_\tau \le \mathbf{E}\, X_s$. Then X is a (right continuous) supermartingale.*

Proof. Since t is a bounded proper stopping time $\ge t$, we have that X_t is integrable for every $t \ge 0$ and X is adapted, so we only need to show that

$$\mathbf{E}\, X_t[\,A\,] \le \mathbf{E}\, X_s[\,A\,]$$

for every $t > s$ and for every $A \in \mathscr{F}_s$. So let's fix $s < t$ and take arbitrary $A \in \mathscr{F}_s$. Let $\tau = s[\,A^C\,] + t[\,A\,]$. Then $s \le \tau \le t$ and moreover, τ is a proper stopping time, since

$$[\,\tau \le u\,] = [\,A\,][\,s \le u < t\,] + [\,u \ge t\,]$$

is $\mathscr{F}_u$-measurable for every fixed u. So the assumption gives that

$$\mathbf{E}\, X_\tau \le \mathbf{E}\, X_s$$

but since $\mathbf{E}\, X_\tau = \mathbf{E}\, X_t[\,A\,] + \mathbf{E}\, X_s[\,A^C\,]$ the claim follows since

$$\mathbf{E}\, X_t[\,A\,] = \mathbf{E}\, X_\tau - \mathbf{E}\, X_s[\,A^C\,] \le \mathbf{E}\, X_s - \mathbf{E}\, X_s[\,A^C\,] = \mathbf{E}\, X_s[\,A\,].$$

□

The Optional Stopping Theorem provides that the converse holds as well i.e. that if we have a right continuous supermartingale, then $\mathbf{E}\, X_\tau \geq \mathbf{E}\, X_\eta$ for every bounded stopping times $\tau \leq \eta$.

First we need a result concerning the adjoining infinities (or limit points) to supermartingales in such a way that we would still have a supermartingale.

5.3. **Theorem.** *If (X_t) is right continuous supermartingale, $T = \mathbb{R}_+$ then the following are equivalent.*

- *i) X_t^- converges in L^1-sense, when $t \to \infty$.*
- *ii) There exists a $X_\infty \in L^1$, such that $X_t^- \leq \mathbf{E}\,(X_\infty^- \mid \mathscr{F}_t)$*
- *iii) The family $\{\, X_t^- :\ t \geq 0 \,\}$ is uniformly integrable.*

If any of these conditions is satisfied, then $X_\infty = \lim_{t\to\infty} X_t$ almost surely. Moreover, if

$$\sup_{t\geq 0} \mathbf{E}\, |X_t|^p < \infty,$$

hold for $p > 1$ then all of the above hold and $X_t^- \to X_\infty^-$ converges in the space L^p as well.[37]

The uniformly integrable family means the following.

5.4. **Definition.** The family $\{\, X_t :\ t \in T \,\}$ of random variables is *uniformly integrable*, if

$$\lim_{M\to\infty} \sup_{t\in T} \mathbf{E}\left([\, |X_t| \geq M \,] |X_t| \right) = 0.$$

Proof. Let's consider just couple of points. *i*) **implies** *ii*): Suppose $X_\infty^- = \lim_t X_t^-$ in L^1, then for every $s > t$ we have

$$\mathbf{E}\,[\,A\,]\mathbf{E}\left(X_\infty^- - X_t^- \mid \mathscr{F}_t\right) = \mathbf{E}\,[\,A\,]\mathbf{E}\left((X_s^- - X_t^- + X_\infty^- - X_s^-) \mid \mathscr{F}_t\right)$$

for every $A \in \mathscr{F}_t$. Since X is supermartingale and the mapping $x \mapsto x^-$ is decreasing and convex, we get by Jensen's inequality that

$$X_t^- \leq (\mathbf{E}\,(X_t \mid \mathscr{F}_t)\,)^- \leq \mathbf{E}\left(X_s^- \mid \mathscr{F}_t\right)$$

and therefore

$$\mathbf{E}\,[\,A\,]\mathbf{E}\left(X_\infty^- - X_t^- \mid \mathscr{F}_t\right) \leq \mathbf{E}\,(X_\infty^- - X_s^-)$$

for every s and the right-hand side goes to 0 by assumption as $s \to \infty$.
The *ii*) **implies** *iii*): The goal is to show

$$\forall \varepsilon > 0 : \exists \alpha > 0 \colon \sup_t \beta_t(\alpha) \leq \varepsilon$$

[37] i.e. there exists a $Z \in L^p$ such that $\|\, X_t - Z \,\|_p^p := \mathbf{E}\, |X_t - Z|^p \to 0$ as $t \to \infty$

where $\beta_t(\alpha) = \mathbf{E}\,[\,A_t(\alpha)\,]X_t^-$ and $A_t(\alpha) = \{X_t^- \geq \alpha\}$. Now

$$X_t^- \leq \mathbf{E}\left(X_\infty^- \,|\, \mathscr{F}_t\right).$$

and therefore, for every $A \in \mathscr{F}_t$ we have

$$\mathbf{E}\,X_t^-[\,A\,] \leq \mathbf{E}\,[\,A\,]\mathbf{E}\left(X_\infty^- \,|\, \mathscr{F}_t\right) = \mathbf{E}\,X_\infty^-[\,A\,].$$

The right-hand side can be estimated by

$$\mathbf{E}\,X_\infty^-[\,A\,] = \mathbf{E}\,X_\infty^-[\,A, X_\infty^- > m\,] + \mathbf{E}\,X_\infty^-[\,A, X_\infty^- > m\,] \leq \varepsilon/2 + m\mathbf{P}\,(\,A\,)$$

when we choose m so large, that $\mathbf{E}\,X_\infty^-[\,X_\infty^- > m\,] < \varepsilon$. In particular,

$$\beta_t(\alpha) \leq \varepsilon + m\mathbf{P}\,(\,A_t(\alpha)\,).$$

By Markov inequality we get

$$\mathbf{P}\,(\,A_t(\alpha)\,) \leq \frac{\mathbf{E}\,X_t^-}{\alpha} \leq \frac{\mathbf{E}\,X_\infty^-}{\alpha} \to 0$$

as $\alpha \to \infty$, so

$$\limsup_{\alpha\to\infty} \sup_t \; \beta_t(\alpha) \leq \varepsilon/2 + \limsup_{\alpha\to\infty} \frac{m\mathbf{E}\,X_\infty^-}{\alpha} \leq \varepsilon$$

and the uniform integrability follows.

The *iii*) **implies** *i*): The uniform integrability gives that there exists an $M > 0$ such that

$$\sup_t \mathbf{E}\,X_t^-[\,X_t^- \geq M\,] \leq 1$$

and therefore,

$$\sup_t \mathbf{E}\,X_t^- \leq \sup_t M\mathbf{P}\left(X_t^- \leq M\right) + \sup_t \mathbf{E}\,X_t^-[\,X_t^- \geq M\,] \leq M + 1 < \infty.$$

The Theorem 3.28 implies that there exists $X_\infty \in L^1$ such that $X_t \to X_\infty$ almost surely, as $t \to \infty$. Therefore, also $X_t^- \to X_\infty^-$ almost surely, since $x \mapsto x^-$ is continuous function.

We leave the proof that almost sure convergence together with the uniform integrability implies the convergence in L^1.

□

5.5. *Remark.* In the previous theorem, if X is a martingale, we see that *iii*) says that $\{\, X_t : \; t \geq 0 \,\}$ is uniformly integrable (not just the negative parts). Furthermore, we see that *ii*) becomes $X_t = \mathbf{E}\,(X_\infty \,|\, \mathscr{F}_t)$, so every X_t is a conditional expectation of a *single* random variable. We call martingale with this property *uniformly integrable* (or UI) and we will call supermartingale with properties *i*)-*iii*) uniformly integrable as well.

For supermartingales we didn't get the full uniform integrability, but we have the following weaker version. Note that if $\tau_n = t_n$ for normal time instances then (t_n) satisfies the integrability and the supermartingale assumptions for the *discrete sampled process.* With the Optional Stopping Theorem (Theorem 5.7) we know that *any* dccreasing sequence (τ_n) in the closed interval $a \le \tau_n \le b$ satisfies rest of the assumptions. But we need this as part of the proof of Optional Stopping Theorem, so we assume those.

5.6. **Lemma.** *Suppose* (X_t) *is (right continuous) supermartingale with respect to to a filtration* $(\mathscr{F}_t)$ *on a* closed *interval* $[a,b]$ *where* $a \ge -\infty$ *and* $b \le \infty$. *Suppose* $(\tau_n)_{n\in\overline{\mathbb{N}}}$ *is an* decreasing sequence *of stopping times* $a \le \tau_n \le b$ *such that* $\forall n \in \overline{\mathbb{N}}\colon X_{\tau_n} \in L^1$ *and*[38] $\forall n \in \overline{\mathbb{N}}, \forall m \ge n\colon X_{\tau_m} \ge \mathbf{E}\left(X_{\tau_n} \,|\, \mathscr{F}_{\tau_m^+}\right)$. *Then* $\{ X_{\tau_n} : \ n \in \mathbb{N} \}$ *is uniformly integrable.*

Proof. Let's denote $\widehat{X}_n := X_{\tau_{-n}}$ and $\mathscr{G}_n = \mathscr{F}_{\tau_{-n}^+}$ for every $-n \in \overline{\mathbb{N}}$. Then $\widehat{X}_n$ is $\mathscr{G}_n$-supermartingale on $-\overline{\mathbb{N}} = \{-\infty, \ldots, -2, -1, 0\}$ since by assumption $\widehat{X}_n$ is integrable and adapted to $\mathscr{G}_n$ and moreover, for every negative integer $m > n$ we have

$$\mathbf{E}\left(\widehat{X}_m \,|\, \mathscr{G}_n\right) = \mathbf{E}\left(X_{\tau_{-m}} \,|\, \mathscr{F}_{\tau_{-n}^+}\right) \le X_{\tau_{-n}} = \widehat{X}_n$$

The idea is to find a positive, integrable and increasing sequence (A_n) on $-\overline{\mathbb{N}}$ such that $Y_n := \widehat{X}_n - A_n$ is a $\mathscr{G}_n$-martingale on $-\overline{\mathbb{N}}$.

Then the Theorem 5.3 implies that $\{Y_n\}_{n\in\overline{-\mathbb{N}}}$ is uniformly integrable, since $Y_n = \mathbf{E}\,(Y_0 \,|\, \mathscr{G}_n)$ for every $n \in -\overline{\mathbb{N}}$. Moreover, since $\mathbf{E}\,A_n \le \mathbf{E}\,A_\infty < \infty$ for every $n \in -\overline{\mathbb{N}}$ and $|A_n| = A_n \ge 0$, we have that $\{A_n\}_{n\in-\overline{\mathbb{N}}}$ is uniformly integrable. Since $\widehat{X}_n = Y_n + A_n$ and sum of two uniformly integrable sequence is uniformly integrable, the claim follows.

So how to construct the sequence (A_n)? Suppose we have already constructed $A_n \ge 0$ and we want to find A_{n+1}. Since $A_n = \widehat{X}_n - Y_n$ and since $(\widehat{X}_n)$ and (Y_n) are $\mathscr{G}_n$-adapted, we know that A_n is $\mathscr{H}$-measurable, for some $\mathscr{H} \subset \mathscr{G}_n$. Since $Y_{n+1} = \widehat{X}_{n+1} - A_{n+1}$ and $Y_n = \mathbf{E}\,(Y_{n+1} \,|\, \mathscr{G}_n)$ we get an equation

$$\widehat{X}_n + A_n = \mathbf{E}\left(\widehat{X}_{n+1} + A_{n+1} \,|\, \mathscr{G}_n\right)$$

or

$$(*) \qquad \mathbf{E}\,(A_{n+1} \,|\, \mathscr{G}_n) = A_n + \mathbf{E}\left(\widehat{X}_n - \widehat{X}_{n+1} \,|\, \mathscr{G}_n\right) \ge 0.$$

If we *demand* that

$$(**) \qquad A_{n+1} = A_n + \mathbf{E}\left(\widehat{X}_n - \widehat{X}_{n+1} \,|\, \mathscr{G}_n\right)$$

[38] by $\overline{\mathbb{N}}$ we mean the set $\mathbb{N} \cup \{\infty\}$

then since the right-hand side of $(**)$ is $\mathscr{G}_n$-measurable, this choice satisfies the identity $(*)$. Moreover, the $A_{n+1} \geq A_n \geq 0$. If we choose $A_{-\infty} = 0$, then we would formally find a suitable sequence by

$$A_n = A_n - A_{-\infty} := \sum_{j=-\infty}^{n-1} \mathbf{E}\left(\widehat{X}_j - \widehat{X}_{j+1} \,|\, \mathscr{G}_j\right)$$

which is clearly increasing in n, positive (since all the terms in sum are) and it is $\mathscr{G}_{n-1}$-measurable for every $n \in \overline{-\mathbb{N}}$. We still need to verify that these sums converge and are integrable. If $n, N \in\leq 0$ and $N < n$, then

$$B_{n,N} := \sum_{j=N}^{n-1} \mathbf{E}\left(\widehat{X}_j - \widehat{X}_{j+1} \,|\, \mathscr{G}_j\right)$$

is a finite sum and $0 \leq B_{n,N} \uparrow A_n$ as $N \downarrow -\infty$. Therefore, by monotone convergence and by *telescoping* sum

$$\mathbf{E}\,A_n = \limsup_{N\to-\infty} \mathbf{E}\,B_{N,n} = \limsup_{N\to-\infty} \mathbf{E}\,\widehat{X}_N - \mathbf{E}\,\widehat{X}_n \leq \mathbf{E}\,\widehat{X}_{-\infty} - \mathbf{E}\,\widehat{X}_0 < \infty$$

So for every $n \leq 0$ the random variables A_n are integrable with uniform integrable upper bound A_0. This shows that we could construct the required sequence (A_n) and the claim follows. □

Now it's time for the Optional Stopping Theorem.

5.7. **Theorem** (Optional Stopping Theorem). *Let (X_t) be a right continuous supermartingale with respect to filtration $(\mathscr{F}_t)$ and suppose that there is an L^1-random variable X_∞, such that*

$$\forall t \geq 0\colon X_t \geq \mathbf{E}\left(X_\infty \,|\, \mathscr{F}_t\right).$$

a) the family $\{\ X_\nu^- :\ \nu$ is a stopping time $\}$ is uniformly integrable and X_τ is integrable for every stopping time τ,
b) $X_\tau[\,\tau = \infty\,] = X_\infty[\,\tau = \infty\,]$ for any stopping time τ
c) If $\tau \leq \eta$ are stopping times, then $X_\tau \geq \mathbf{E}\left(X_\eta \,|\, \mathscr{F}_{\tau+}\right)$

If X is a martingale, then there is an equality in c).

5.8. *Remark.* If (X_t) is a right continuous submartingale with respect to filtration $(\mathscr{F}_t)$ and if there is an L^1-random variable Y such that

$$\forall t \geq 0\colon X_t \leq \mathbf{E}\left(Y \,|\, \mathscr{F}_t\right),$$

then the same conclusions for X hold where in d) the inequality is reversed, i.e. $X_\tau \leq \mathbf{E}\left(X_\eta \,|\, \mathscr{F}_{\tau+}\right)$.

Proof. Let's sketch the main parts of the proof.

Suppose τ and η are simple stopping times i.e. takes only finitely many values (and potentially value ∞) and assume that $\tau \leq \eta$. Then clearly X_η and X_τ are integrable since for instance

$$\mathbf{E}\,|X_\tau| \leq \sum_d \,[\,\tau = d\,]\,\mathbf{E}\,|X_d| < \infty.$$

The remaining part, namely that X_τ is $\mathscr{F}_\tau = \mathscr{F}_{\tau^+}$-measurable and $X_\tau \geq \mathbf{E}\,(X_\eta \,|\, \mathscr{F}_{\tau^+})$ is left to excercises.

τ and η any stopping times. Let's define

$$\tau_n := 2^{-n}(\lfloor 2^n \tau \rfloor + 1)[\,\tau < n\,] + \infty[\,\tau \geq n\,]$$
$$\eta_n := 2^{-n}(\lfloor 2^n \eta \rfloor + 1)[\,\eta < n\,] + \infty[\,\eta \geq n\,]$$

These are simple and proper stopping times. Moreover, $\tau_n \downarrow \tau$ and $\eta_n \downarrow \eta$ almost surely and thus, by right continuity, $X_{\tau_n} \to X_\tau$ and $X_{\eta_n} \to X_\eta$ almost surely.

By the previous part for simple stopping times, we know that $X(\tau_n)$ is integrable for every n and $\mathbf{E}\,(X(\tau_{n+1}) \,|\, \mathscr{F}_{\tau_n}) \leq X(\tau_n)$. Hence we know that by Lemma 5.6 that $\{X_{\tau_n}\}$ is uniformly integrable. Similarly $\{X_{\eta_n}\}$ is uniformly integrable and therefore, we have $X_{\tau_n} \to X_\tau$ and $X_{\eta_n} \to X_\eta$ in L^1-sense as well. Thus, both X_τ and X_η are integrable, and moreover for every $A \in \mathscr{F}_{\tau^+} = \bigcap \mathscr{F}_{\tau_n}$ and therefore, we have

$$\mathbf{E}\,[\,A, \tau < \infty\,](X_{\tau_n} - X_{\eta_n}) = \sum_{d \in \mathcal{D}} \mathbf{E}\,[\,A, \tau_n = d\,]\mathbf{E}\,(X_{\tau_n} - X_{\eta_n} \,|\, \mathscr{F}_{\tau_n}) \geq 0$$

for[39] every $n \in \mathbb{N}$. Since we know that $X_{\eta_n} \to X_\eta$ in and $X_{\tau_n} \to X_\tau$ in L^1, we can move the limits inside and obtain

$$\mathbf{E}\,[\,A, \tau < \infty\,](X_\tau - X_\eta) \geq 0$$

for every $A \in \mathscr{F}_{\tau^+}$. When $\tau = \infty$, we have $\eta \geq \tau = \infty$ and also $\tau_n = \eta_n = \infty$ and so $X_\tau = X_\eta = X_\infty$. Thus trivially

$$\mathbf{E}\,[\,A, \tau = \infty\,](X_\tau - X_\eta) = 0$$

So we have shown part of *a*) and *c*). Part *b*) is part of the construction.

So only thing remaining is to show the uniform integrability. Since now we know that $X_\tau \geq \mathbf{E}\,(X_\infty \,|\, \mathscr{F}_{\tau^+})$, since $\eta = \infty$ is a stopping time. Therefore, like in the proof of Theorem 5.3 we deduce that $X_\tau^- \leq \mathbf{E}\,(X_\infty^- \,|\, \mathscr{F}_{\tau^+})$. And like

[39]the set $\mathcal{D}$ is the dyadic rationals, and for more detailed argument you are advised to look at the proof of the Theorem 4.21

in the proof of Theorem 5.3 we can now conclude that for every $\varepsilon > 0$ there exists an $m > 0$ such that for every $\alpha > 0$ we have

$$\sup_{\tau} \mathbf{E}\, X_\tau^- [\, X_\tau^- > \alpha\,] \leq \varepsilon + m \sup_{\tau} \mathbf{P}\left(X_\tau^- > \alpha \right) \leq \varepsilon + \frac{m\mathbf{E}\, X_\infty^-}{\alpha}$$

□

5.1. **Local martingales.** Next we will define the *local martingale.* Localisation always refers to some kind of cutting or breaking into small pieces. With the Optional Stopping Theorem, we can show that stopped martingale is a martingale. To give meaning to this let us define

$$X^\tau(t) := X(t \wedge \tau)$$

where τ is a stopping time. By inspection, we notice that after time τ this process *stops* and stays in the state $X(\tau)$. This does not destroy the martingale property.

5.9. **Lemma.** *Suppose X is right continuous supermartingale with respect to filtration $(\mathscr{F}_t)$. Then the stopped process $Y := X^\tau$ is a right continuous supermartingale with respect to filtration $(\mathscr{F}_t)$ suhteen for every bounded stopping time τ.*

Proof. Let $s \geq 0$ and $\eta \geq s$ be a bounded stopping time. Then $Y(\eta) = X(\eta \wedge \tau)$, and therefore by the Optional Stopping Theorem, $\mathbf{E}\,|Y(\eta)| < \infty$. Since X is a supermartingale we have by the Optional Stopping Theorem, that

$$\mathbf{E}\, X(\eta \wedge \tau) \leq \mathbf{E}\, X(s \wedge \tau)$$

or in other words $\mathbf{E}\, Y_\eta \leq \mathbf{E}\, Y_s$. This, however, by the Lemma 5.2 implies that Y is a supermartingale. □

Since the martingale property is preserved while stopping the following generalizes the concept of a martingale.

5.10. **Definition.** Let (X_t) be a right continuous process and adapted to a filtration $(\mathscr{F}_t)$. We say that X is a *local martingale* with respect to to filtration $(\mathscr{F}_t)$ if there exists a sequence (τ_n) of stopping times such that $\tau_n \uparrow \infty$ and $X^{\tau_n}[\,\tau_n > 0\,]$ is uniformly integrable martingale for every n.

5.11. *Remark.* In the same way we can define *local supermartingales*, *locally bounded processes* etc.

5.12. *Remark.* While we are at it let's generalize martingales to vector processes by saying that the $\mathbb{R}^d$-valued process is a martingale with respect to to $(\mathscr{F}_t)$ if each of its coordinates are $(\mathscr{F}_t)$-martingales.

5.13. **Example.** Every *right continuous* martingale is a local martingale. Just take $\tau_n = n$, since then X^{τ_n} is right continuous and uniformly integrable.

We can still define local processes on random time intervals.

5.14. **Definition.** Let (X_t) be an $(\mathscr{F}_t)$-adapted right continuous process. Let τ be a $(\mathscr{F}_t)$-stopping time. We will say that X is a *local martingale on the random interval* $[0,\tau)$ with respect to filtration $(\mathscr{F}_t)$ if there exists a sequence (τ_n) of stopping times such that $\tau_n \uparrow \tau$ and $X^{\tau_n}[\,\tau_n > 0\,]$ is uniformly integrable $(\mathscr{F}_t)$-martingale for every n.

5.2. **Hunt processes and Debut Theorem.** Let's return to Feller processes. We know that Brownian motion is a Feller process and that it is continuous. We have almost every time assumed the right continuity from the processes at hand but for Feller processes that is not really needed.

5.15. **Theorem.** *Suppose X is a Feller process. Then it has a* càdlàg *version i.e. there is a Feller process X' that is right continuous and has left limits at every time instances and X' is a version*[40] *of X for every starting point $x \in S$.*

Proof. We will postpone this proof. □

In addition to this, the Feller processes have a weak continuity from left, namely the quasi-left continuity.

5.16. **Definition.** Let $\mathcal{M} = (X_t, \mathscr{F}_t, \mathbf{P}_x)$ be a Markov process. We say that X is *quasi-left continuous*, if

$$\mathbf{P}_x\left(\lim_{n\to\infty} X_{\tau_n} = X_\tau, \tau < \infty\right) = \mathbf{P}_x\left(\,\tau < \infty\,\right)$$

for every *stopping time* τ and every increasing sequence of *stopping times* (τ_n) such that $\tau_n \uparrow \tau$, we have for every $x \in S$.

5.17. **Theorem.** *A càdlàg Feller process is quasi-left continuous.*

Proof. Suppose first that $\tau < \infty$. Since X has left limits, we can define a random variable $Y = \lim X_{\tau_n}$. The goal is to prove that $\mathbf{P}_x\left(\,Y = X_\tau\,\right) = \mathbf{E}_x\left[\,Y = X_\tau\,\right] = 1$ for every x. If we can show that $\mathbf{E}_x\, u(Y, X_\tau) = \mathbf{E}_x\, u(Y,Y)$ for every bounded and measurable $u\colon S\times S \to \mathbb{R}$, the claim follows by choosing $u(x,y) = [\,x = y\,]$.

Since in S we can approximate indicator functions of rectangles by functions of form $f(x)g(x)$ where f and g are $C_\infty(S)$-functions, the claim follows by π-λ Theorem, if we can show that $\mathbf{E}_x\, f(Y)g(X_\tau) = \mathbf{E}_x\, f(Y)g(Y)$.

[40] so we have $\mathbf{P}_x\left(\,X'_t = X_t\,\right) = 1$ for every t and for every $x \in S$

Let $t > 0$ and let's rewrite $\mathbf{E}_x\, f(Y)g(X_\tau)$ as

$$(*)\qquad \begin{aligned} \mathbf{E}_x\, f(Y)g(X_\tau) &= \mathbf{E}_x \left(\lim_{n\to\infty} f(X_{\tau_n})g(X_\tau)\right) \\ &= \lim_{n\to\infty} \mathbf{E}_x\, f(X_{\tau_n})(g(X_{\tau_n+t}) + g(X_\tau) - g(X_{\tau_n+t}))\,. \end{aligned}$$

The first term on the right of $(*)$ can be dealt with the strong Markov property at τ_n:

$$\mathbf{E}_x\, f(X_{\tau_n})g(X_{\tau_n+t}) = \mathbf{E}_x\, f(X_{\tau_n})\mathbf{E}_{X_{\tau_n}}\, g(X_t) = \mathbf{E}_x\, f(X_{\tau_n})P_t g(X_{\tau_n})\,.$$

Since $g \in C_\infty(S)$ the Feller property shows that $P_t g \in C_\infty(S)$ and so we can compute the limit

$$\lim_{n\to\infty} \mathbf{E}_x\, f(X_{\tau_n})g(X_{\tau_n+t}) = \lim_{n\to\infty} \mathbf{E}_x\, f(X_{\tau_n})P_t g(X_{\tau_n}) = \mathbf{E}_x\, f(Y)P_t g(Y)\,.$$

Furthermore, if we let $t \downarrow 0$, the Feller property says that $t \mapsto P_t g$ is continuous in the uniform norm, so we have that

$$\lim_{t\downarrow 0}\lim_{n\to\infty} \mathbf{E}_x\, f(X_{\tau_n})g(X_{\tau_n+t}) = \mathbf{E}_x\, f(Y)g(Y)\,.$$

So we only have to show that the second term of $(*)$ vanishes at the limit, i.e. we need that

$$(**)\qquad \lim_{t\downarrow 0}\lim_{n\to\infty} \mathbf{E}_x\, f(X_{\tau_n})(g(X_\tau) - g(X_{\tau_n+t})) = 0.$$

We can estimate that

$$\mathbf{E}_x\, |f(X_{\tau_n})(g(X_\tau) - g(X_{\tau_n+t}))| \le \|\, f\,\|_\infty \mathbf{E}_x \sup_{s\in(\tau_n-\tau,t)} |g(X_\tau) - g(X_{\tau+s})|$$

By dominated convergence and the fact that $\tau_n \to \tau$ we get

$$\limsup_{n\to\infty} \mathbf{E}_x\, |f(X_{\tau_n})(g(X_\tau) - g(X_{\tau_n+t}))| \lesssim \mathbf{E}_x \sup_{s\in[0,t)} |g(X_\tau) - g(X_{\tau+s})|$$

Since X is right continuous and g is continuous,

$$\lim_{t\downarrow 0} \sup_{s\in[0,t)} |g(X_\tau) - g(X_{\tau+s})| = 0$$

almost surely. Thus, we get claimed $(**)$ by dominated convergence. □

The following theorem provides a lot of stopping times we need when analysing the stochastic processes on domains, namely the hitting and the debut times.

5.18. **Definition.** Let X be a stochastic process with state space $(S, \mathscr{B}(S))$. Let $G \in \mathscr{B}(S)$. We define the *first hitting time* τ_G to a set G as

$$\tau_G := \inf\{\, t > 0 :\ X(t) \in G\,\}$$

and the *first entrance time* D_G into the set G as

$$D_G := \inf\{\, t \ge 0 :\ X(t) \in G\,\}.$$

By a *first exit time from a set* G we mean the first hitting hitting time to G^C.

5.19. *Remark.* These time are clearly connected but might be different. If $X_0 \notin G$, then $\tau_G = D_G$. Moreover, in general

$$\{D_G \leq t\} = \{X_0 \in G\} \cup \{\tau_G \leq t, X_0 \notin G\}$$

5.20. **Theorem** (Debut Theorem)**.** *Suppose* (X_t) *is right continuous and quasi-left continuous strong Markov process with respect to to filtration* $(\mathscr{F}_t)$*. Suppose* $S \subset \mathbb{R}^d$ *is locally compact Polish space.*

a) *When* G *is an open set, then* τ_G *and* D_G *are* $(\mathscr{F}_{t^+})$*-stopping times.*
b) *When* G *is a closed set, then* τ_G *and* D_G *are* $(\mathscr{F}_{t^+})$*-stopping times.*
c) *When* $G \subset S$ *is a Borel set, then* τ_G *and* D_G *are are* $(\widetilde{\mathscr{F}_t})$*-stopping times.*

Proof. *a*) G **is open.** We will first show τ_G is a stopping time and deduce D_G from this.

Suppose $\tau_G < t$. Then by definition of τ_G there exists an $s < t$ such that $X_s \in G$. Since G is open and X is right continuous, there is a $u \in (s,t)$, such that $X_v \in G$ for every $v \in [s,u]$. In particular, there is a rational $r \in [s,u] \subset [0,t)$ such that $X_r \in G$.

On the other hand, if there is a rational $r < t$ such that $X_r \in G$, we have that $\tau_G < t$.

Therefore, we have deduced that

$$\{\tau_G < t\} \; = \bigcup_{r \in \mathbb{Q}} \{r < t, X_r \in G\} \in \mathscr{F}_t^0 \subset \mathscr{F}_t$$

which shows that τ_G is $(\mathscr{F}_{t^+})$-stopping time. Here $(\mathscr{F}_t^0)$ is the *history* of X and $\mathscr{F}_t^0 \subset \mathscr{F}_t$ since X is adapted to $\mathscr{F}_t$. From this we deduce that the first entrance time D_G is also a stopping time, since

$$\{D_G \leq t\} \; = \{X_0 \in G\} \; \cup \{X_0 \notin G, \tau_G \leq t\} \; \in \mathscr{F}_{t^+}^0 \subset \mathscr{F}_{t^+}.$$

b) G **is closed.**

Since S is a Polish space, we can find a decreasing sequence of *open sets* U_n such that

$$\forall n \colon U_n \supset \overline{U}_{n+1} \; \supset G.$$

and that

$$G = \bigcap_n \overline{U}_n = \bigcap_n U_n.$$

By *a*) the sequence $(\tau_n') := (\tau_{U_n})$ are stopping times. Furthermore, the sequence of stopping times $\tau_n' := \tau_{U_n}$ satisfies

$$\tau_1' \leq \tau_2' \leq \dots \tau_n' \leq \dots \leq \tau_G$$

since if $\tau'_n(\omega) \ = t$, then $X_s(\omega) \notin U_n$ for every $s < t$ and so $X_s(\omega) \notin U_{n+1}$ for every $s < t$ which implies that $\tau'_{n+1}(\omega) \geq t = \tau'_n(\omega)$. In the same way we can deduce that $\tau_G \geq \tau'_n$ for every n.

Thus, we can define a limit stopping time $\nu = \lim_n \tau'_n$ which is satisfies $\nu \leq \tau_G$. By quasi-left continuity, we know that

$$X_\nu = \lim_{n\to\infty} X_{\tau'_n}$$

from which we can deduce that $X_\nu \in G$ (as we soon see)

Therefore, by the definition of τ_G we have $\nu \geq \tau_G$ if we assume that $X_0 \notin G$ because if $X_0 \in G$ then by construction $\nu = 0$ but it might be that $\tau_G > 0$ (we will see this in excercises later on). But comparing the above to D_G we notice that without assuming $X_0 \notin G$ we have $\nu = D_G$.

However, since ν is a stopping time, we obtain that D_G is a $(\mathscr{F}_{t^+})$-stopping time but also $(\mathscr{F}^0_{t^+})$-stopping time as well. To deduce that τ_G is a stopping time, we can continue as follows. If $\tau_G = 0$, then $D_G = 0$ and so $\{\tau_G = 0\} \ \in \mathscr{F}_{0^+}$. This means that if for every $t > 0$ and every $n \in \mathbb{N}_+$ we have

$$\{1/n < \tau_G \leq t\} \ \in \mathscr{F}_{t^+}$$

we are done. For $\widetilde{X_s} := X_{s+1/n}$ we have that $\widetilde{X_0} \notin G$ and so the first hitting time $\widetilde{\tau}_G$ and first entrance time $\widetilde{D_G}$ for $\widetilde{X}$ coincide i.e. $\widetilde{\tau}_G = \widetilde{D_G} = \tau_G - 1/n$. This means that

$$\{0 < \widetilde{\tau_G} \leq t - 1/n\} \ \in \sigma \bigcup \{\ \widetilde{X_s} :\ s \in [0, t-1/n]\ \} \subset \mathscr{F}^0_{t^+} \subset \mathscr{F}_{t^+}$$

but since $\{1/n < \tau_G \leq t\} \ = \{0 < \widetilde{\tau_G} \leq t - 1/n\}$ we are done.

□

We will later on deal processes that generalize the Feller processes a bit, namely they satisfy the properties of Theorem 5.15 and Theorem 5.17. So we end this chapter with a preliminary definition of the Hunt processes.

5.21. **Preliminary definition.** A Markov process $\mathcal{M} = (X_t, \mathscr{F}_t, \mathbf{P}_x, P_{t,x})$ with a transition function $P_{t,x}$, a *compact state space* $S^\dagger = S \cup \{\dagger\}$ and a *completed filtration* $\mathscr{F}_t = \widetilde{\mathscr{F}_t}$ is called a *Hunt process*, if and only if

- *i*) it is right continuous,
- *ii*) it has the strong Markov property
- *iii*) and it is quasi-left continuous

Therefore, every Feller process is a Hunt process (by Theorem 5.17). Moreover, we could prove that almost surely every path of a Hunt process has left limits, i.e. almost every path is càdlàg.

5.3. **Some examples.**

5.22. **Example.** We have said a couple of times that the Brownian motion lives in the whole space. Let's put some meat on it. This could be setteled by answering the following questions:

Q1. Let $x_0, x \in \mathbb{R}^d$. Will Brownian motion always hit the point x when $B_0 = x_0$?

Q2. Let $x_0, x \in \mathbb{R}^d$ and $r > 0$. Will Brownian motion always hit the the ball of radius r and centered at the point x when $B_0 = x_0$?

Q3. Let $x_0 \in \mathbb{R}^d$, and $r > 0$. Will Brownian motion always hit the complement of the ball of radius r and centered at the origin ?

We see that Q1 > Q2 > Q3, if we interpret this to mean: "Q1 > Q2 if a positive answer to $Q1$ answers $Q2$ positively." We can rephrase all of these as the question of finiteness of the hitting time τ_G.

Q1. Let $x_0, x \in \mathbb{R}^d$. Is $\mathbf{P}_{x_0}\left(\tau_{\{x\}} < \infty\right) = 1$?

Q2. Let $x_0, x \in \mathbb{R}^d$ and $r > 0$. Is $\mathbf{P}_{x_0}\left(\tau_{B_r(x)} < \infty\right) = 1$?

Q3. Let $x_0 \in \mathbb{R}^d$ and $r > 0$. Is $\mathbf{P}_{x_0}\left(\tau_{B_r(0)^C} < \infty\right) = 1$?

Furthermore, in 1-dimensional case at least $Q1$ and $Q2$ are the same since Brownian motion is continuous.

In higher dimensional case the questions are not necessarily the same. So let's consider the Q3 and start from a point $x \in B_r(0)$. Let's denote the stopping time simply by τ. If we draw a box $[-r, r]^d$ around the ball and call the hitting time to the complement of this box by η, we deduce

$$\mathbf{P}_x\left(\eta < \infty\right) \le \mathbf{P}_x\left(\tau < \infty\right).$$

Moreover, since hitting the boundary of the box means hitting one of the edges we have

$$\begin{aligned}\mathbf{P}_x\left(\eta < \infty\right) &= \mathbf{P}_x\left(\eta^{(1)} < \infty\right) + \mathbf{P}_x\left(\eta^{(1)} = \infty, \eta^{(2)} < \infty\right) + \dots \\ &\ge \mathbf{P}_x\left(\eta^{(1)} < \infty\right).\end{aligned}$$

Therefore, if the question Q3 can be answered positively in 1-dimensional case, we have a positive answer to the question Q3 in every dimensions.

Let's show that $Q3$ in 1-dimensions hold by showing that even

$$\mathbf{E}_x\,\eta^{(1)} = \lim_{n\to\infty} \mathbf{E}_x\left(\eta^{(1)} \wedge n\right) < \infty.$$

And again, let's just denote this stopping time as τ. For this we will use the Optional Stopping Theorem for a uniformly integrable martingale. We know couple of martingales by now, namely the Brownian motion B_t and the martingale $X_t = B_t^2 - t$. Neither of these are uniformly integrable, *but by stopping* we can produce uniformly integrable martingales from these.

First we notice that

$$Z_t = X_{t\wedge n}$$

is a martingale and since $Z_t = \mathbf{E}\,(Z_n \,|\, \mathscr{F}_t) = \mathbf{E}\,(Z_\infty \,|\, \mathscr{F}_t)$ it is uniformly integrable by Theorem 5.3. Therefore, Z_τ is integrable and also

$$\mathbf{E}_x\, Z_0 = \mathbf{E}_x\, Z_\tau$$

by the Optional Stopping Theorem. Now $Z_0 = X_0 = B_0^2$, so

$$\mathbf{E}_x\, Z_0 = \mathbf{E}_x\, B_0^2 = x^2.$$

The right-hand side is

$$\mathbf{E}_x\, Z_0 = \mathbf{E}_x\,(B_{\tau\wedge n}^2 - (\tau \wedge n))$$

so we obtain an equation

$$\mathbf{E}_x\,(\tau \wedge n) = \mathbf{E}_x\, B_{\tau\wedge n}^2 - x^2$$

for every n. If the right-hand side is uniformly bounded with respect to n, then we obtain $\mathbf{P}_x\,(\,\tau < \infty\,) = 1$, since

$$n\mathbf{P}_x\,(\,\tau = \infty\,) = \mathbf{E}_x\,[\,\tau = \infty\,](\tau \wedge n) \le \mathbf{E}_x\,(\tau \wedge n)\,.$$

However, this is seen easily, since $\tau \wedge n \le \tau$. Therefore, $|B_{\tau\wedge n}| \le r$, since τ is the first time the absolute value of Brownian motion reaches the level r. This, however, means that the sequence $(B_{\tau\wedge n}^2)$ has an integrable upper bound r^2. So, we first obtain that

$$\mathbf{P}_x\,(\,\tau = \infty\,) \le n^{-1}\mathbf{E}_x\,(\tau \wedge n) \le n^{-1}(r^2 - x^2) \to 0$$

as $n \to \infty$ and so we deduce $\mathbf{P}_x\,(\,\tau = \infty\,) = 0$. Furthermore, this means that $\tau \wedge n \to \tau < \infty$ almost surely and hence $B_{\tau\wedge n}^2 \to B_\tau^2 = r^2$ almost surely by the continuity of Brownian motion and by the property that $B(\tau_G) \in \overline{G}$ for every Borel set G.

Note, that the finiteness of τ is needed here, since B_∞ cannot be defined. But once we have deduced the finiteness, we get the almost sure convergence and since we had the integrable upper bound, we can use the dominated convergence and deduce

$$\mathbf{E}_x\,\tau = \lim_{n\to\infty} \mathbf{E}_x\,(\tau \wedge n) = \mathbf{E}_x\, B_\tau^2 - x^2 = r^2 - x^2$$

This means that we have settled the question Q3 in every dimension.

Let's still consider Q1 in 1-dimensional case (the other can be dealt with the stochastic integration). So let's do the same with procedure with Brownian motion B instead of X, but this time with the stopping time $\tau_{-r} \ \wedge \tau_r$ where τ_a is the first time when $B_t = a$.

This (as in excercises) leads to an identity

$$x = \mathbf{E}_x\, B_0 \ = \mathbf{E}_x\, B_{\tau_r \wedge \tau_{-r}} \ = r\mathbf{P}_x\,(\,\tau_r < \tau_{-r}\,) - r\mathbf{P}_x\,(\,\tau_{-r} < \tau_r\,)$$

for every $x \in (-r, r)$. Since Brownian motion is continuous we have

$$\mathbf{P}_x\,(\,\tau_{-r} < \tau_r\,) + \mathbf{P}_x\,(\,\tau_{-r} > \tau_r\,) = 1 - \mathbf{P}_x\,(\,\tau_{-r} = \tau_r\,) = 1 - \mathbf{P}_x\,(\,\tau_{-r} = \tau_r = \infty\,)$$

and if $\tau_r = \tau_{-r} = \infty$, then $\tau = \infty$ which we know to to an almost impossible. So

$$\mathbf{P}_x\,(\,\tau_{-r} < \tau_r\,) + \mathbf{P}_x\,(\,\tau_{-r} > \tau_r\,) = 1$$

and we obtain two linear equations for the two probabilities and thus, we can solve it leading to

$$\mathbf{P}_x\,(\,\tau_{-r} < \tau_r\,) = \frac{x+r}{2r}, \quad \text{and} \quad \mathbf{P}_x\,(\,\tau_{-r} > \tau_r\,) = \frac{r-x}{2r}.$$

If we change the coordinates so that x becomes $r_N := N - (r - x)$, the upper limit r becomes N and therefore, $-r$ becomes $-N$ then we get that

$$\mathbf{P}_x\,(\,\tau_r < \tau_{-r_N}\,) = \mathbf{P}_{N-(r-x)}\,(\,\tau_N < \tau_{-N}\,) = \frac{2N - (r-x)}{2N} \to 1$$

as $N \to \infty$. We notice that $\tau_{-r_N} \uparrow \infty$, since they form an increasing sequence and if it has a finite limit, then Brownian motion would reach $-\infty$ at bounded time which is not possible by continuity. Therefore, we get from the monotone convergence that

$$\mathbf{P}_x\,(\,\tau_r < \infty\,) = \lim_{N\to\infty} \ \mathbf{P}_x\,(\,\tau_r < \tau_{-r_N}\,) = 1.$$

However, in this case the expectation of τ_r is infinite, since we saw above that $\tau_r \wedge \tau_{-r_N} \uparrow \tau_r$ and so

$$\mathbf{E}_x\, \tau_r \ = \lim_{N\to\infty} \mathbf{E}_x\, \tau_r \wedge \tau_{-r_N} \ = \lim_{N\to\infty} \mathbf{E}_{N-(r-x)}\,(\tau_N \wedge \tau_{-N})$$

Since $\tau_N \wedge \tau_{-N}$ is the first exit time from the interval $(-N, N)$ i.e. the Q3, we can compute the expectation on the right and it is

$$\mathbf{E}_{N-(r-x)}\,(\tau_N \wedge \tau_{-N}) \ = N^2 - (N - (r-x))^2 = 2N(r-x) - (r-x)^2 \to \infty$$

as $N \to \infty$.