4. Feller processes and strong Markov processes

4.1. **Feller processes.** Let's return to Brownian motion and its properties. We already know that it is a Markov process, and we have considered its transition probability operators $(P_t^{(B)})$. These happen to be very nice operators since they preserve continuous functions and we will give a name for the Markov processes that posess this property.

These Markov processes are going to be called *Feller processes* and Brownian motion (again) is one of those.

In the sequel we will need a function space, that consists of those scalar valued functions on the state space that are continuous and that *vanish at infinity*. We will denote this set by $C_\infty(S)$. In order to discuss about the infinity [27], we will require from this point on that the state space S is compact or at least locally compact. Later on this $S^\dagger$ will be this *compact* state space. So we define $C_\infty(S) = \{ f \in C(S^\dagger) : \ f(\dagger) = 0 \}$.

4.1. **Definition.** Suppose that S is compact or locally compact Polish space.[28] A semigroup[29] (T_t) has a *Feller property*, if $T_t f \in C_\infty(S)$ for every $f \in C_\infty(S)$ and in addition, it satisfies,

$$\lim_{t \to 0} \| T_t f - f \|_\infty = 0$$

for every $f \in C_\infty(S)$. A Markov process (X_t) is called a *Feller process*, if its transition probability operators have a Feller property.

4.2. *Remark.* Note that the condition on the state space limits the S as a *subset of* $\mathbb{R}^d$ somewhat. Since compact is locally compact, every open subset of $\mathbb{R}^d$ is still fine as well as well as every closed subset of $\mathbb{R}^d$. However, this does leave out some of the Borel sets already, since only sets that are countable intersections of open sets satisfy the assumption of being a Polish space in the relative topology and the subsets that are locally compact in $\mathbb{R}^d$ are precisely intersections of an open and a closed set.[30]

Therefore, we can read that the assumption we made for the state space is that as a subset of $\mathbb{R}^d$ it must be an intersection of an open and a closed set.

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[27]We will interpret infinity to mean a one point compactification $\dagger$. If the state space is compact, we add an *isolated point*

[28]i.e. it is separable and has a complete metric

[29]recall this means that we assume that $T_0 = \mathrm{I}$ and $T_t T_s = T_{t+s}$ for every $s, t \geq 0$ and that $0 \leq u \leq v$ implies that $0 \leq T_t u \leq T_t v$ for bounded measurable functions

[30]James Dugunji: Topology, Theorem 6.3. page 239

We have nearly shown already that Brownian motion is a Feller process, since we know that $P_t^{(B)}$ maps the space[31] $L^\infty(S)$ to $C_b(S)$. We will still need to verify that $P_t^{(B)} f$ vanishes at infinity as long as $f \in C_\infty(S)$.

4.3. **Lemma.** *Brownian motion is a Feller process.*

Proof. Let $f \in C_\infty(\mathbb{R}^d)$ and $\varepsilon > 0$. We know that $P_t^{(B)} f$ that is even $C^\infty(\mathbb{R}^d)$-function. So we only need to show that

$$P_t^{(B)} f(x) = \int_{\mathbb{R}^d} p_t(x,y) f(y) \,\mathrm{d}y$$

häviää, kun $|x| \to \infty$. Since $f \in C_\infty(\mathbb{R}^d)$, we can choose $R_0 > 0$ in such a way that $[\, B_R^C \,] f \leq \varepsilon$ for every $R > R_0$. Therefore,

$$|P_t^{(B)} f(x)| \leq \int_{\mathbb{R}^d} p_t(x,y) [\, |y| < R \,] f(y) \,\mathrm{d}y + \varepsilon$$

for every $R > R_0$ and every $x \in \mathbb{R}^d$. When $|x| \geq 2R$, we can estimate that [32], $p_t(x,y) \lesssim e^{-R^2/2t}$, and so

$$\sup_{|x| \geq 2R} |P_t^{(B)} f(x)| \leq cR^d \| \, f \, \|_\infty e^{-R^2/2t} + \varepsilon$$

The claim follows from this since we may choose large enough $R > R_0$, such that $cR^d \| \, f \, \|_\infty e^{-R^2/2t} \leq \varepsilon$. By the definition of the limit

$$\lim_{|x| \to \infty} P_t^{(B)} f(x) = 0$$

or in other words $P_t^{(B)} f \in C_\infty(\mathbb{R}^d)$. □

If we are given a submarkovian semigroup (T_t) with Feller property, we can define a transition function $(P_{t,x})$ so that

$$T_t f(x) = \int_S f(y) P_{t,x}(\,\mathrm{d}y)$$

for every $f \in C_\infty(S)$. This means that for every Feller semigroup there exists a corresponding Markov process!

4.4. **Lemma.** *For every submarkovian semigroup (T_t) with Feller property, there exists a unique transition function $(P_{t,x})$ such that*

$$T_t f(x) = \int_S f(y) P_{t,x}(\,\mathrm{d}y)$$

for every $f \in C_\infty(S)$ and every $x \in S$.

[31] This should be read as the bounded measurable functions, even though this is not entirely correct

[32] usually while estimating we are not interested in the actual constants so we will be using the notation $\lesssim$ to describe that the use of *implicit constants*

Proof. We notice that for every $x \in S$ and for every $t \geq 0$ the map $f \mapsto T_t f(x)$ is a positive linear functional on $C_\infty(S)$. Since S is a locally compact, the Riesz Representation Theorem says that it corresponds to unique measure, namely

$$T_t f(x) = \int_S f(y) P_{t,x}(\,\mathrm{d}y)$$

for every $f \in C_\infty(S)$. We next verify that $(P_{t,x})$ is a transition function. First of all, since

$$\int_S f(y) P_{0,x}(\,\mathrm{d}y) = T_0 f(x) = f(x)$$

for every $f \in C_\infty(S)$, then if U is a finite intersection of balls[33] and it is a relatively compact set (let's denote the family of these as $\mathscr{C}$), we can find a monotone sequence $f_n \uparrow [U]$ of $C_\infty(S)$-functions (why? excercise), and so by the monotone convergence theorem

$$P_{0,x}(U) = \lim_{n\to\infty} \int_S f_n(y) P_{0,x}(\,\mathrm{d}y) = \lim_{n\to\infty} f_n(x) = [\, x \in U\,]$$

Now the set $\mathscr{G}_1$ of U's for which

$$P_{0,x}(U) = [\, x \in U\,]$$

holds forms a Dynkin system, and since finite intersections of balls is a π-system, the $\mathscr{G}_1 = \mathscr{S} = \mathscr{B}(S)$. Also, we notice that since $x \mapsto T_t f(x)$ is continuous, it is Borel measurable for every f. Again, this means that

$$x \mapsto P_{t,x}(U) = \lim_{n\to\infty} T_t f_n(x)$$

is measurable for every $U \in \mathscr{C}$. Denoting by $\mathscr{G}_2$ the sets for which the measurability holds, we obtain a Dynkin system and Dynkin's π-λ again gives that $\mathscr{G}_2 = \mathscr{B}(S)$. Last thing to check is the Chapman–Kolmogorov equation. Again we have for every $f \in C_\infty(S)$ that

$$\int_S P_{t+s,x}(\,\mathrm{d}y) f(y) = T_{t+s} f(x) = T_t T_s f(x) = \int_S P_{t,x}(\,\mathrm{d}y) T_s f(y)$$
$$= \int_S P_{t,x}(\,\mathrm{d}y) \int_S P_{s,y}(\,\mathrm{d}z) f(z)$$

The monotone convergence theorem with $[\,U\,]$ for $U \in \mathscr{C}$ gives then

$$P_{t+s,x}(U) = \int_S P_{t,x}(\,\mathrm{d}y) P_{s,y}(U)$$

and the standard π-λ techinque proves the claim. □

[33]by a ball we mean a set $V_x(r) = \{\, y : \ d(x,y) < r \,\}$

4.2. **Stopping time.** The one of the first goals we have is to show the Kakutani's result and because of that we want to compute the expectation

$$u(x) = \mathbf{E}_x\, f(B(\tau))\,,$$

where τ is the *first exit time* of Brownian motion from the domain G. So let's now define the concept of *stopping time*, which generalizes the exit time, as we will see later. The following discrete time definition sheds light on the more general definition.

4.5. **Preliminary definition.** When $T = \mathbb{N}$ and S is countable, the random variable $\tau\colon \Omega \to T$ is a *stopping time* with respect to the filtration $(\mathscr{F}_n)$, if

$$\{\tau = n\} \in \mathscr{F}_n$$

for every $n \in \mathbb{N}$.

This generealizes the concept of *absorption time* for Markov chains, which is the first time a chain enters an absorbing state in which case the chain gets stuck (*gets absorbed*). This is a random time which we can say that if it has happened at the given time or not.

So in general stopping time means that we are waiting for some phenomenon to happen and we can think that there is an apparatus that will indicate this, say, by going from *waiting state* to *stopped state*. If we know the whole history at the current time, we can with certainty say, if the apparatus is still in a waiting state or in a stopped state. This is what the condition $\{\tau = n\}\ \in \mathscr{F}_n$ is describing.

For the continuous time case the border between the presetn and the past is somewhat vague, so we have no reason to assume that $\{\tau = t\}$ would not be a null event[34].

We notice, however, that for the discrete time case τ is a stopping time if and only if $\{\tau \leq n\} \in \mathscr{F}_n$ for every $n \in \mathbb{N}$ (Excercise).

This generalizes now with ease and therefore

4.6. **Definition.** Let $(\mathscr{F}_t)$ be a filtration. We say that a random variable $\tau\colon \Omega \to T$ is a *stopping time* with respect to the filtration $(\mathscr{F}_t)$, if

$$\{\tau \leq t\} \in \mathscr{F}_t$$

for every $t \in T$. We will also call it $(\mathscr{F}_t)$-stopping time.

Before listing properties of stopping times we need the concept of right continuous filtration.

[34]i.e. an almost impossible event

4.7. **Definition.** Let $(\mathscr{F}_t)$ be a filtration. We denote

$$\mathscr{F}_{t^+} := \sigma \bigcap_{s>t} \mathscr{F}_s.$$

In the sequel we will call $(\mathscr{F}_t)$-stopping times as *proper stopping times* and $(\mathscr{F}_{t^+})$-stopping times shortly as *stopping times*. We also say that $(\mathscr{F}_t)$ is *right continuous*, if for every $t \geq 0$ it holds that $\mathscr{F}_t = \mathscr{F}_{t^+}$.

Note that every proper stopping time is a stopping time and if the filtration is right continuous these concepts are the same.

4.8. *Remark.* If $(\mathscr{F}_t)$ is a filtration, then the filtration $(\mathscr{F}_t^+)$ is right continuous. (Excercise).

Now we list some properties (and leave the proofs to excercises) of stopping times (and proper stopping times). Let's explicitly write out the filtration so that one can see what hold for proper stopping times and what just for the stopping times.

4.9. **Example.** Let (τ_n) be a sequence of $(\mathscr{F}_t)$-stopping times (i.e. proper stopping times). Then the following hold.[35]

(1) Suppose $\tau = t$ is a constant random variable. Then τ is a $(\mathscr{F}_t)$-stopping time.
(2) $\tau_1 \wedge \tau_2$ is a $(\mathscr{F}_t)$-stopping time.
(3) $\tau_1 \vee \tau_2$ is a $(\mathscr{F}_t)$-stopping time.
(4) $\tau_1 + \tau_2$ is a $(\mathscr{F}_t)$-stopping time.
(5) $\tau := \sup \tau_n$ is a $(\mathscr{F}_t)$-stopping time.
(6) τ_1 is also a $(\mathscr{F}_{t^+})$-stopping time.
(7) $\tau := \inf \tau_n$ is a $(\mathscr{F}_{t^+})$-stopping time.
(8) $\tau := \limsup \tau_n$ is a $(\mathscr{F}_{t^+})$-stopping time.
(9) $\tau := \liminf \tau_n$ is a $(\mathscr{F}_{t^+})$-stopping time.
(10) if the limit $\tau := \lim \tau_n$ exists, then it is a $(\mathscr{F}_{t^+})$-stopping time.

4.10. *Remark.* We notice that for right continuous $(\mathscr{F}_t)$, all the above are (proper) stopping times. This is the reason that we will usually *assume that the filtrations are right continuous*, but do we know, for instance, that Brownian motion is Markov process and a martingale with respect to the right continuous history.

It is good to note how the stopping times and proper stopping times differ.

[35]We denote $x \wedge y := \min(x, y)$ and $x \vee y := \max(x, y)$.

4.11. **Lemma.** *A random variable τ is a $(\mathscr{F}_{t+})$-stopping time if and only if for every $t > 0$ it holds that $\{\tau < t\} \in \mathscr{F}_t$.*

Proof. Let's leave this to excercises. □

The right continuity of the filtration is more or less about null sets i.e. about the sets whose probability or a measure is zero. Since these are the obstacles, we can bypass them by *just adding the null sets to the filtration from the start*!

4.12. **Definition.** Let μ be a probability measure on a measurable space $(\Omega, \mathscr{F})$. We will denote the completed σ-algebra

$$\mathscr{G}^\mu := \sigma\left(\mathscr{G}, \mathcal{N}_\mu\right)$$

which is obtained from a given σ-algebra $\mathscr{G}$ by adding the μ-null sets $\mathcal{N}_\mu$ to it. *The completed σ-algebra* is

$$\widetilde{\mathscr{G}} := \sigma\left(\bigcap_\mu \mathscr{G}^\mu\right).$$

Let's list (without proofs), few things connected with completions:

4.13. **Proposition.** *Let $(\mathscr{F}_t)$ be a filtration. Then $(\widetilde{\mathscr{F}_t})$ and $(\mathscr{F}_t^\mu)$ for every μ are right continuous.*

Proof. Omitted. I will add this later to Appendices. □

4.3. **Strong Markov property for Feller processes.** Now we can finally introduce the Markov property that really separates the discrete and continuous time cases from each other. In the discrete case every Markov chain posesses the *strong Markov property*, but for the continuous case this is no longer true in general. But, as always, Brownian motion has this property as well.

In a discrete time case with countable state space, i.e. when a Markov process (X_n) is a Markov chain and τ is a *stopping time* we can easily describe the *history* $\mathscr{H}_\tau$ at the stopping time τ by defining

$$\mathscr{H}_\tau := \bigcup_{n,(i_j)} \{\tau = n, X_0 = i_0, \ldots, X_n = i_n\}$$

which is the family of all the paths of the chain that end at the stopping time τ given $\tau = n$. Again, this is cannot be generalized immediately, but we notice that we can express this σ-algebra in another way:

$$\mathscr{H}_\tau = \{\ A : \ \{A \text{ ja } \tau \le n\} \in \mathscr{H}_n \text{ for every } n \in \mathbb{N}\ \}.$$

This formulation for the history at stopping time τ can now be generalized so we define

4.14. **Definition.** Let $(\mathscr{F}_t)$ be a filtration and let τ be a $(\mathscr{F}_t)$-stopping time. Then *the σ-algebra $\mathscr{F}_\tau$* at the stopping time τ is the family of those events $A \in \mathscr{F}$ that

$$\mathscr{F}_\tau = \{\, A \in \mathscr{F} :\ \forall t \in T\colon \{A, \tau \le t\} \in \mathscr{F}_t \,\}.$$

Analogously we define

$$\mathscr{F}_{\tau^+} = \{\, A \in \mathscr{F} :\ \forall t \in T\colon \{A, \tau \le t\} \in \mathscr{F}_{t^+} \,\}.$$

kun τ on $(\mathscr{F}_{t^+})$-stopping time.

Now let's list few simple properties of σ-algebras at stopping times.

4.15. **Example.** The families $\mathscr{F}_\tau$ and $\mathscr{F}_{\tau^+}$ have the following basic properties.

(1) Both $\mathscr{F}_\tau$ and $\mathscr{F}_{\tau^+}$ are σ-algebras.
(2) We have

$$\mathscr{F}_{\tau^+} = \{\, A \in \mathscr{F} :\ \forall t \in T\colon \{A, \tau < t\} \in \mathscr{F}_t \,\}.$$

(3) If $(\mathscr{F}_t)$ is right continuous, then $\mathscr{F}_\tau = \mathscr{F}_{\tau^+}$ for every stopping time τ.
(4) If $\tau = t$, then $\mathscr{F}_\tau = \mathscr{F}_t$ and $\mathscr{F}_{\tau^+} = \mathscr{F}_{t^+}$.
(5) If τ is a (proper) stopping time, then τ is $\mathscr{F}_{\tau^+}$-measurable.
(6) If τ_1, τ_2 are (proper) stopping times and $\tau_1 \le \tau_2$, then $\mathscr{F}_{\tau_1^+} \subset \mathscr{F}_{\tau_2^+}$.
(7) If τ_n is a *decreasing sequence* of stopping times and $\tau = \lim_n \tau_n$, then

$$\mathscr{F}_{\tau^+} = \sigma \bigcap_{n=1}^{\infty} \mathscr{F}_{\tau_n^+}.$$

If in addition, every τ_n is propert stopping time, then we have

$$\mathscr{F}_{\tau^+} = \sigma \bigcap_{n=1}^{\infty} \mathscr{F}_{\tau_n}$$

Now we can define the strong Markov property.

4.16. **Definition.** A stochastic process (X_t) has a *strong Markov property with respect to filtration* $(\mathscr{F}_t)$, if for every time instance $t < \infty$, for every $x \in S$ and for every $A \in \mathscr{S}$ it holds that

$$[\,\tau < \infty\,]\mathbf{P}_x\left(X_{t+\tau} \in A \,|\, \mathscr{F}_\tau \right) = [\,\tau < \infty\,]\mathbf{P}_x\left(X_{t+\tau} \in A \,|\, X_\tau \right) \quad \text{a.s.} \tag{4.17}$$

The time stationary strong Markov property means that

$$[\,\tau < \infty\,]\mathbf{P}\left(X_{t+\tau} \in A \,|\, \mathscr{F}_\tau \right) = [\,\tau < \infty\,]\mathbf{P}_{X_\tau}\left(X_t \in A \right) \quad \text{a.s.} \tag{4.18}$$

4.19. *Remark.* As with the Markov property, we will only be treating the time stationary case so we will omit saying time stationary in the sequel.

We will now show that the Brownian motion has the strong Markov property. Actually, we show much more, namely that (right continuous) Feller processes have strong Markov property.

4.20. *Remark.* Recall that S is locally compact Polish space and $S^\dagger$ is the one-point compactification of S. If S is is compact, then $S^\dagger = S \cup \{\dagger\}$ where $\dagger$ is an isolated point. Thus, $C_\infty(S) = \{\ f \in C(S^\dagger) :\ f(\dagger) = 0\ \}$ is really just $C_b(S)$. When S is not compact, then $f \in C(S^\dagger)$ means that $f = f(\dagger) + g$ with $g \in C_\infty(S)$.

4.21. **Theorem.** *Let (X_t) be a right continuous Feller process. Then for every $(\mathscr{F}_{t+})$-stopping time τ and for every $t \geq 0$*

$$[\,\tau < \infty\,]\mathbf{E}_x(f(X_{\tau+t})\,|\,\mathscr{F}_{\tau^+})\ = [\,\tau < \infty\,]P_t^{(X)} f(X_\tau) \quad \text{a.s.}$$

for every $f \in C(S^\dagger)$.

Proof. Let us define $\tau_n := 2^{-n}(\lfloor 2^n \tau \rfloor + 1)$. This implies that $\tau_n \downarrow \tau$ and moreover, τ_n is a proper stopping time for every n, since

$$[\,\tau_n \leq t\,] = \sum_{k+1 \leq 2^n t} [\,\tau_n 2^n = k+1\,] = \sum_{k+1 \leq 2^n t} [\,k \leq \tau 2^n < k+1\,]$$

is $\mathscr{F}_t$-measurable. Therefore,

$$\mathscr{F}_{\tau^+} = \sigma \bigcap_{n=1}^{\infty} \mathscr{F}_{\tau_n}.$$

Furthermore, we know that for each fixed $n \in \mathbb{N}$, the random variable τ_n only takes countable many values. Actually, we can say that $\{\forall n \colon \tau_n \in \mathcal{D}\} = \{\tau < \infty\}$ where $\mathcal{D} = \{\ 2^{-n}k :\ n \in \mathbb{N}, k \in \mathbb{Z}\ \}$ is the countable set of dyadic rational numbers. In other words, we can write

$$[\,A, \tau < \infty\,]\ = \sum_{d \in \mathcal{D}} [\,A, \tau_n = d\,].$$

If we combine these two properties, we can conclude that $A \in \mathscr{F}_{\tau^+}$ if and only if $A \cap \{\tau_n = d\} \in \mathscr{F}_d$ for every $d \in \mathcal{D}$ and for every $n \in \mathbb{N}$.

Therefore,

$$\begin{aligned}
\mathbf{E}_x\, f(X_{\tau_n+t})[\,A, \tau < \infty\,] &= \sum_{d \in \mathcal{D}} \mathbf{E}_x\, f(X_{\tau_n+t})[\,A, \tau_n = d\,] \\
&= \sum_{d \in \mathcal{D}} \mathbf{E}_x\, \mathbf{E}_x(f(X_{d+t})\,|\,\mathscr{F}_d)\ [\,A, \tau_n = d\,] \\
&= \sum_{d \in \mathcal{D}} \mathbf{E}_x\, P_t^{(X)} f(X_d)[\,A, \tau_n = d\,]
\end{aligned}$$

where the last identity is the usual Markov property. By going backwards, we can continue and we get rid of the sum

$$\begin{aligned}\mathbf{E}_x\, f(X_{\tau_n+t})[\,A,\tau<\infty\,] &= \sum_{d\in\mathcal{D}} \mathbf{E}_x\, P_t^{(X)} f(X_d)[\,A,\tau_n=d\,] \\ &= \sum_{d\in\mathcal{D}} \mathbf{E}_x\, P_t^{(X)} f(X_{\tau_n})[\,A,\tau_n=d\,] \\ &= \mathbf{E}_x\, P_t^{(X)} f(X_{\tau_n})[\,A,\tau<\infty\,]\,.\end{aligned}$$

We are almost done now, since let's assume in addition[36] that $f\in C_\infty(S)$. Then by Feller property $P_t^{(X)} f\in C_\infty(S)$ for every $t<\infty$. Letting $n\to\infty$ and using the Lebesgue dominated convergence theorem, we first deduce that

$$\mathbf{E}_x \lim_{n\to\infty} f(X_{\tau_n+t})[\,A,\tau<\infty\,] = \mathbf{E}_x \lim_{n\to\infty} P_t^{(X)} f(X_{\tau_n})[\,A,\tau<\infty\,]\,.$$

Since f and $P_t^{(X)} f$ are continuous, we get

$$\mathbf{E}_x\, f(\lim_{n\to\infty} X_{\tau_n+t})[\,A,\tau<\infty\,] = \mathbf{E}_x\, P_t^{(X)} f(\lim_{n\to\infty} X_{\tau_n})[\,A,\tau<\infty\,]$$

and by right continuity of the paths of X and the fact that $\tau_n\downarrow\tau$, we obtain

$$\mathbf{E}_x\, f(X_{\tau+t})[\,A,\tau<\infty\,] = \mathbf{E}_x\, P_t^{(X)} f(X_\tau)[\,A,\tau<\infty\,]$$

which is precisely what we wanted to show.

In the general case when $f\in C(S^\dagger)$ the function $f_0(x)=f(x)-f(\dagger)$ is in $C_\infty(S)$. Thus,

$$[\,\tau<\infty\,](\mathbf{E}_x(f(X_{\tau+t})-f(\dagger)\,|\,\mathscr{F}_{\tau^+})\,) = [\,\tau<\infty\,](P_t^{(X)} f(X_\tau)-f(\dagger)) \quad \text{a.s.}$$

for every $f\in C(S^\dagger)$. Since the conditional expectation of a constant is the same constant, the claim follows. □

4.22. *Remark.* Note that if we assume that always $X_\infty(\omega)=\dagger$, then

$$[\,\tau=\infty\,]\mathbf{E}_x(f(X_{\tau+t})\,|\,\mathscr{F}_{\tau^+}) = [\,\tau=\infty\,]f(\dagger) = [\,\tau=\infty\,]P_t^{(X)} f(\tau).$$

Thus, if we assume that at infinity X is in the state $\dagger$, we can express the strong Markov property *without always stating that* $\tau<\infty$.

[36]if S is compact, this is no addition at all, this is only extra information for local compact case