2. Brownian motion

We will now define the Brownian motion. We will see eventually few different kinds of definitions for the same concept.

2.1. **Preliminary definition.** *1-dimensional Brownian motion* $\{B(t)\}_{t\geq 0}$ is *a Gaussian process*, with *the mean* $\mathbf{E}\,B\,(t) = 0$ and the *covariance function* $\mathbf{E}\,B\,(t)B(s) = \min(t,s)$.

Recall that the process is called *Gaussian*, if its every finite sample is normally distributed random variable. In other words, if for every $n \in \mathbb{N}_+$ and every $t_1 < \cdots < t_n$, the random variable $(B(t_1),\ldots,B(t_n))$ is n-dimensional Gaussian random variable. From this simple preliminary definition we can deduce a lot of the properties of the process (naturally, since this is (nearly) equivalent with any other definition)

2.2. **Lemma.** 1*-dimensional Brownian motion has the following properties:*

i) *it has identically distributed increments, i.e. the process* $X(t) = B(t+h) - B(h)$ *is a Brownian motion for every* $h > 0$.

ii) *it has independent increments, i.e. for every* $n \in \mathbb{N}_+$ *and every* $t_1 < \cdots < t_n$ *the family* $\{B(t_2)-B(t_1),\ldots,B(t_n)-B(t_{n-1})\}$ *is an independent family of random variables.*

iii) *the variance of an increment is*

$$\mathbf{V}\,\left(X(t)-X(s)\right) = \mathbf{V}\,X(t-s) = |t-s|$$

iv) *the increments satisfy for every* $N \in \mathbb{N}$*:*

$$\mathbf{E}\,|X(t)-X(s)|^{2N} \leq \gamma_N |t-s|^N .$$

v) *it has a* Hölder continuous[13] *version and any* $\alpha < \frac{1}{2}$ *can be used as its Hölder exponent.*

vi) *Brownian motion exists and we may assume that it has (Hölder) continuous paths.*

Proof. The properties *i)*, *ii)*, *iii)* and *iv)* are left as for excercises and we comment on properties *v)* and *vi)* below. □

The property *vi)* follows from *v)* and Andrey Nikolaevich Kolmogorov's Extension Theorem, that says, that to show the existence of a process it is enough to construct the finite marginal in a consistent way.

[13] a mapping $f\colon S \to \mathbb{R}$ is Hölder continuous with exponent $\alpha > 0$ on a normed space S, if $\|\, f(x) - f(y)\,\| \leq c|x-y|^\alpha$

Theorem (Kolmogorov's Extension Theorem). *Let T be a set of times and $S \subset \mathbb{R}^d$ is a Borel subset. Then the process $(X(t); t \in T)$ exists if and only if for every* finite subset $F \subset T$ *the restricted stochastic process $(X^F(t); t \in F)$ exists and if $F' = F \cup \{t\}$ then $(X^{F'}(t); t \in F)$ is identically distributed with $(X^F(t); t \in F)$.*

Proof. This is omitted. □

2.3. *Remark.* Let's think what the theorem says. If $(X(t))$ exists then naturally the restricting the set of times does not change the existence so the one direction is simple. The extension part (from finite to infinite) is more demanding. When the set of times $T_1 = \{t_1, \ldots, t_n\}$ is finite, the existence of the random variable is equivalent with the existence of a n-dimensional random variable $(X(t_1), \ldots, X(t_n))$. This might depend of T_1, so we denote this potential dependence by writing $(X^{T_1}(t))$. If we enlarge the set of times with one time instance t, so $T_2 = \{t_1, \ldots, t_n, t\}$ and the X^{T_2} corresponds to $n+1$-dimension random variable $(X^{T_2}(t_1), \ldots, X^{T_2}(t_n), X(t))$. The condition that the stochastic process $(X^{T_2}(t))$ restricted to T_1 is identically distributed then means that

$$\mathbf{P}\left(X^{T_1}(t_j) \in A_j,\ \forall j = 1, \ldots, n \right) = \mathbf{P}\left(X^{T_2}(t_j) \in A_j,\ \forall j = 1, \ldots, n \right)$$

Most of the time the set of times is *ordered* i.e. $t_1 < \cdots < t_n$. Then adding one time instance might go to $n+1$ different places, i.e. $t < t_1$, or $t_1 < t < t_2$, and so on. Usually we express this *consistency condition* in terms of the ordered marginal distributions

$$\begin{aligned} &\mu^{T_2}_{(t_1,\ldots,t_n,n+1)}(A_1, \ldots, A_k, S, A_{k+2}, \ldots, A_n) \\ &= \mu^{T_1}_{(t_1,\ldots,t_k,t_{k+2},t_{n+1})}(A_1, \ldots, A_k, A_{k+2}, \ldots, A_n) \end{aligned}$$

The assumption that the state space S is a Borel subset (with $\mathscr{S} = \mathscr{B}(S)$) is not necessary but sufficient. The "real" requirement is that the probability measures μ^{T_1} are always *inner regular*.[14] The space with this property is called a Radon space and if we assume that S is a Borel subset, then $(S^N, \mathscr{S}(S^N))$ is a Radon space for every N.

The properties $i)$, $ii)$ and $iii)$ of the Brownian motion easily provide the finite dimensional marginals. Let's construct the distribution of $(B(t_1), B(t_2), B(t_3))$, where $0 < t_1 < t_2 < t_3$, as an example. We will determine the conditional

[14] A measure μ is inner regular, if $\mu(B)$ of any Borel set B is the supremum (i.e. be approximated from inside) of $\mu(K)$ for compact sets $K \subset B$.

expectation

$$\begin{aligned}
&\mathbf{P}\left(B_{t_3} \in \mathrm{d}x_3 \,|\, B(t_1), B(t_2) \right) \\
&= \mathbf{P}\left(B_{t_3} - B_{t_2} \in \mathrm{d}x_3 - B(t_2) \,|\, B(t_1), B(t_2) \right) \\
&= \mathbf{P}\left(B_{t_3} - B_{t_2} \in \mathrm{d}x_3 - x_2 \,|\, B(t_1), B(t_2) \right) |_{x_2 = B(t_2)} \\
&= \mathbf{P}\left(B_{t_3} - B_{t_2} \in \mathrm{d}x_3 - x_2 \right) |_{x_2 = B(t_2)} \\
&= q_{t_3 - t_2}(x_3 - B_{t_2})\,\mathrm{d}x_3,
\end{aligned}$$

where f_t is the *density function* of the normally distributed random variable $\mathcal{N} \sim \mathfrak{N}(0,t)$. In the second identity we used the fact, that in the measurable part can be considered known in the conditional expectation and in the third we used the definition of independence. From this we can deduce further that

$$\begin{aligned}
&\mathbf{P}\left(B(t_2) \in \mathrm{d}x_2, B_{t_3} \in \mathrm{d}x_3 \,|\, B(t_1) \right) \\
&= \mathbf{E}\left(\mathbf{P}\left(B(t_2) \in \mathrm{d}x_2, B_{t_3} \in \mathrm{d}x_3 \,|\, B(t_2), B(t_1) \right) \,|\, B(t_1)\right) \\
&= \mathbf{E}\left(q_{t_3-t_2}(x_3 - B(t_2))[\, B(t_2) \in \mathrm{d}x_2 \,] \,|\, B(t_1)\right)\ \mathrm{d}x_3 \\
&= q_{t_3-t_2}(x_3 - x_2)\mathbf{P}\left(B(t_2) \in \mathrm{d}x_2 \,|\, B(t_1) \right)\,\mathrm{d}x_3 \\
&= q_{t_3-t_2}(x_3 - x_2) q_{t_2-t_1}(x_2 - B_{t_1})\,\mathrm{d}x_2\,\mathrm{d}x_3,
\end{aligned}$$

where we repeat the same deduction again and we conditionalised inside a conditional expectation. This then leads to the formula for the distribution

$$\begin{aligned}
&\mathbf{P}\left(B(t_1) \in \mathrm{d}x_1, B(t_2) \in \mathrm{d}x_2, B_{t_3} \in \mathrm{d}x_3 \right) \\
&= q_{t_3-t_2}(x_3 - x_2) q_{t_2-t_1}(x_2 - x_1) q_{t_1}(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3.
\end{aligned}$$

We will return to this computation soon when we start to study *Markov processes.*

Property v) follows from another theorem of Kolmogorov, namely from the Kolmogorov Continuity Theorem. The following is a very simplified version, but that's good enough for us.

Theorem (Kolmogorov Continuity Theorem)**.** *Suppose that state space $S \subset \mathbb{R}^d$ is open or closed and set of times $T = [a,b]$ is a closed interval. Let $(X(t); t \in T)$ be a stochastic process. If we can find such* strictly positive *constants α, β and γ that*

$$\mathbf{E}\,|X(t) - X(s)|^{\alpha} \le \gamma |t-s|^{1+\beta},$$

then the process $(X(t))$ has a version $(\widetilde{X}(t))$*, whose almost all paths satisfy*

$$|\widetilde{X}(t) - \widetilde{X}(s)| \le C|t-s|^{r/\alpha}$$

for every $t, s \in [a,b]$ and for every $r \in (0,\beta)$. Furthermore, if X itself is continuous, then X itself satisfies the Hölder condition above.

Proof. Omitted. □

Since we now know that Brownian motion can be assumed to behave nicely, it is reasonable to add this to *definition* to begin with.

2.4. **Definition.** A 1-dimensional Brownian motion $(B(t); t \geq 0)$ is *continuous* Gaussian process, that satisfies the following conditions:

(1) $\mathbf{E}\, B(t) = 0$ and $\mathbf{V}\, B(t) = t$ for every $t \geq 0$
(2) if $t_1 < \cdots < t_n$ and $h > 0$, then the distribution of the random variable

$$(B(t_2 + h) - B(t_1 + h), \ldots, B(t_n + h) - B(t_{n-1} + h))$$

does not depend on the value of the parameter h.
(3) if $t_1 < \cdots < t_n$, then the random variables $\{B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})\}$ are independent.

The property (2) above is called *stationary increments* and the property (3) is called *independent increments*. We can now simply generalize this to higher dimensions and we can deduce the existence and the properties of the higher dimensional Brownian motion from the existence and properties of 1-dimensional Brownian motion.

2.5. **Definition.** Let $d \in \mathbb{N}_+$. A d-dimensional Brownian motion $(B(t); t \geq 0) = ((B^{(1)}(t), \ldots, B^{(d)}(t)); t \geq 0)$ is a continuous d-dimensional Gaussian process, that satisfies the following conditions:

(1) for every $j = 1, \ldots, d$ coordinate process $B^{(j)}$ is a Brownian motion
(2) processes $B^{(j)} \perp\!\!\!\perp B^{(k)}$ when $j \neq k$.

This definition implies that Brownian motion "starts" from the origin. We can generalize it to *start from any given point* $x \in \mathbb{R}^d$ by defining:

2.6. **Definition.** Let $x \in \mathbb{R}^d$. We say that a process $X(t)$ is a *Brownian motion starting from point* x, if $X(t) - x$ is a Brownian motion.

We also notice that the probability for the Brownian motion that starts from the origin to be at the time instance $t > 0$ *outside* a ball $B_R(0)$ of radius R is at least

$$\mathbf{P}_0\left(|B_t| > R \right) \geq \left(2\mathbf{P}\left(\sqrt{t}\mathcal{N} > R\right)\right)^d > 0,$$

where $\mathcal{N} \sim \mathfrak{N}(0,1)$. The estimate follows, since a ball of radius R fits inside a cube with side lenght $2R$. Since the coordinate Brownian motions are independent, we can reduce the computation of the probability to the distribution of a real-valued standard Gaussian distributed random variable $\mathcal{N}$.

So it is possible, that the Brownian motion does not stay inside of any bounded domain, and we shall soon see that it is *almost sure*[15], that it leaves any domain with enough time or in other words, with probability 1 there is a $t > 0$, such that $|B_t| > R$.

Thus, if we want to stay inside of a given domain D, we must use some *other process* than the Brownian motion.

[15]remember, almost sure means with probability 1