11. Reflecting Brownian motion in higher dimensions

The construction of the 1-dimensional reflecting Brownian motion provides two methods for constructing the reflecting Brownian motion in $G \subset \mathbb{R}^d$.

The first method is to generalize the Skorohod equation and use that to construct reflecting Brownian motion R in G. This we will cover in more detail.

The other method is more analytical. We construct a Markov process X which we then show to coincide in law with R by showing that both of these solve a Submartingale Problem of Stroock and Varadhan. This we will also cover in detail.

11.1. **Generalized Skorohod's equation.** Suppose we want to construct a reflecting Brownian motion in a domain $G \subset \mathbb{R}^d$ with sufficiently smooth boundary so that we can easily define the outer unit normal $\nu\colon \partial G \to \mathbb{R}^d$ to every point $x \in \partial G$. For instance, if G is a unit ball, then we have $\nu(x) = x$ for every $x \in \partial G$ and if $G = \mathbb{R}^{d-1} \times \mathbb{R}_+$ is the half space, then $\nu(x) = (0, \ldots, 0, -1)$ gives the outer unit normal.

11.1. **Definition.** Suppose y is a path in $\mathbb{R}^d$ and $y(0) \in \overline{G}$. The pair (ξ, L) is called the solution of the *Skorohod problem* SE(y, G) *for* y *and* G, if (ξ, L) have the following properties.

i) The $t \mapsto \xi(t)$ is a path in $\overline{G}$.

ii) The $t \mapsto L(t)$ is a nondecreasing function, and which increases only when $\xi \in \partial G$, i.e.

$$L(t) = \int_0^t [\, \xi(s) \in \partial G \,] \, \mathrm{d}L(s)$$

iii) the functions ξ, L and y together solve the Skorohod's equation

$$\xi(t) = y(t) - \tfrac{1}{2} \int_0^t \nu(\xi(s)) \, \mathrm{d}L(s)$$

If ξ and L are continuous, we will call the solution continuous. We also write $(\xi, L) = (\xi', L')$ for two solutions, if $\xi = \xi'$ and $L = L'$.

We notice that this is a straight forward generalization of the 1-dimensional Skorohod's equation. This equation can be solved in Lipschitz domains but we will consider only C^2-domains.

11.2. **Theorem** (Uniqueness for SE(y, G))**.** *Suppose G has a C^2-boundary and suppose y is a continuous path. If (ξ, L) and (ξ', L') are two continuous solution to* SE(y, G)*, then $(\xi, L) = (\xi', L')$.*

11.3. **Theorem** (Existence for $\mathrm{SE}(y, G)$)**.** *Suppose G has a C^1-boundary and suppose y is a continuous path. Then there exists a continuous solution (ξ, L) to the Skorohod problem* $\mathrm{SE}(y, G)$.

Once we have these, we can apply that to Brownian motion i.e. solve for every path of Brownian motion the Skorohod problem $\mathrm{SE}(B.(\omega), G)$ thus obtaining a unique solution $(R(\omega), L(\omega))$. It is easy to check that both R and L are processes that are adapted to the filtration $(\mathscr{F}_t^B)$.

11.4. **Definition.** We say that an process R is a reflecting Brownian motion in G starting at $x \in \overline{G}$, if it has the same law as the solution of the Skorohod problem $\mathrm{SE}(B., G)$ for Brownian motion starting at $x \in \overline{G}$.

We would like to show that reflecting Brownian motion is a strong Markov process. For this we can use the Submartingale Problem of Stroock and Varadhan.

11.2. **Submartingale problem.** The following problem was introduced by Daniel Stroock and Srinivasa Varadhan in 1971. Let's first introduce the measurable space of continuous functions.

11.5. **Definition.** We will denote by $\widetilde{\Omega}$ the set of continuous functions $\widetilde{\omega}$ that map $[0, \infty)$ to $\mathbb{R}^d$.

We also define $\widetilde{X}_t(\widetilde{\omega}) := \widetilde{\omega}(t)$ which defines a stochastic process on $(\widetilde{\Omega}, \mathscr{M}^0)$ when we define a σ-algebra $\mathscr{M}^{t_0} := \sigma(\{\, \widetilde{X}_t : \ t \geq t_0 \,\})$.

The process $\widetilde{X}$ generates a natural filtration $(\mathscr{M}_t^{t_0})_{t \geq t_0}$ on Ω by $\mathscr{M}_t^{t_0} := \sigma(\{\, \widetilde{X}_s : \ t_0 \leq s \leq t \,\})$.

The Submartingale Problem is closely related with the connection we observed while we considered the Dirichlet problem and the Kakutani's representation theorem. Let's first formulate the special case of the Submartingale Problem and after we have seen how it connects to the reflecting Brownian motion, we can formulate the Submartingale Problem in general.

We start by defining a partial differential operator $L_t(a, b)$. Here $a = (t, x) \mapsto a(t, x)$ is a matrix valued function and $b = (t, x) \mapsto b(t, x)$ is a vector valued function. For the reflecting Brownian motion these coefficients are the identity matrix $a(x, t) = I$ and zero vector $b(t, x) = 0$. In this case, the partial differential operator $L_t(I, 0) f(t, x) = \frac{1}{2} \triangle_x f(t, x)$.

The reflection on the boundary corresponds in the Submartingale Problem to a partial differential operator $J_t(\gamma, \rho)$ on the boundary. In the reflecting Brownian motion case the coefficients $\gamma(x, t) = -\nu(x)$ for every $x \in \partial G$ and the coefficient $\rho(x,t) = 0$. In this case, the operator becomes $J_t(-\nu,0)f(t,x) = -\partial_{\nu(x)} f(t,x) = -\nu(x)\cdot\nabla f(t,x)$ for every $x \in \partial G$.

Now we can introduce the Submartingale Problem.

11.6. **Definition.** We say that a probability measure $\mathbf{P}$ on the measurable space $(\widetilde{\Omega}, \mathscr{M}^{t_0})$ solves the *Submartingale Problem* sMP(a,b,γ,ρ,G) *on the domain G for coefficients a, b, γ and ρ* if it has the following two properties:

(A) $\forall t \geq t_0 \colon \mathbf{P}(\widetilde{X}_t \in \overline{G}) = 1$ and

(B) $\forall f \in C_0^{1,2}([0,\infty)\times\mathbb{R}^d)$ that satisfy $J_t(\gamma,\rho)f(t,x) \geq 0$ on $[t_0,\infty)\times\partial G$ the process

$$Z_t := f(t,\widetilde{X}_t) - \int_{t_0}^{t} [\,\widetilde{X}_u \in G\,](\partial_u f + L_u f)(u,\widetilde{X}_u)\,\mathrm{d}u$$

is a $(\mathscr{M}_t^{t_0})$-submartingale with respect to the probability measure $\mathbf{P}$.

The solutions of the Submartingale Problem have many nice properties. One is the uniqueness of the solution to the problem. The following result is from Stroock and Varadhan 1971.

11.7. **Theorem** (Uniqueness for time-independent coefficients, Stroock–Varadhan 1971)**.** *If the coefficients a, b, γ and ρ don't depend on time, and G has C^2-boundary, then the following conditions are sufficient for the uniqueness of the solution of* sMP(a,b,γ,ρ,G).

i) the function $x \mapsto a(x)$ is continuous and for every $x \in \overline{G}$ the matrix $a(x)$ is symmetric and positive definite (i.e. all the eigenvalues are strictly positive)

ii) the function $x \mapsto b(x)$ is bounded and measurable

iii) the function $x \mapsto \gamma(x)$ is bounded, locally Lipschitz and there exists a $\beta \geq 0$ such that for every $x \in \partial G$ the inner product $-\gamma(x)\cdot\nu(x) \geq \beta$ i.e. the vector $\gamma(x)$ points strictly inside G.

iv) the function ρ is bounded, continuous and non-negative.

When a solution to the Submartingale Problem sMP(a,b,γ,ρ,G) is unique then the process $\widetilde{X}$ is a strong Markov process.

11.8. **Theorem** (Strong Markov property and sMP, Stroock–Varadhan 1971)**.** *Suppose the Submartingale Problem* sMP(a,b,γ,ρ,G) *has a unique solution* $\mathbf{P}$. *Then $\{\widetilde{X}_t, \mathscr{M}_t^{t_0}, \{\mathbf{P}_x\}\}$ is strong Markov process on the measurable space $(\widetilde{\Omega}, \mathscr{M}^{t_0})$ where $\mathbf{P}_x\,(\,A\,) = \mathbf{P}\left(\,A \mid \widetilde{X}_0 = x\,\right)$.*

How do we benefit from these results? Well, if we show that the reflecting Brownian motion defined via Skorohod problem SE$(B.,G)$ *generates* a probability measure $\mathbf{P}$ that solves the Submartingale Problem sMP$(I,0,-\nu,0,G)$.

Then we already know that it is a strong Markov process with continuous paths, i.e. it is a diffusion process.

11.9. **Theorem.** *Suppose R is a reflecting Brownian motion. Then* $\widetilde{\mathbf{P}}(A) := \mathbf{P}_x\,(\,R. \in A\,)$ *solves the Submartingale Problem* $\mathrm{sMP}(I, 0, -\nu, 0, G)$ *for every* $x \in \overline{G}$.

Proof. We first note that $\mathbf{P}_x\,(\,R. \in A\,) = \widetilde{\mathbf{P}}(\widetilde{X}. \in A)$. This is since $\widetilde{X}.(\widetilde{\omega}) = t \mapsto \widetilde{X}_t(\widetilde{\omega}) = t \mapsto \widetilde{\omega}(t) = \widetilde{\omega}$ and thus, $\widetilde{X}.(\widetilde{\omega}) \in A$ if and only if $\widetilde{\omega} \in A$. This means that

$$\widetilde{\mathbf{P}}(\widetilde{X}. \in A) = \widetilde{\mathbf{P}}(A) = \mathbf{P}_x\,(\,R. \in A\,) \tag{11.10}$$

The relation (11.10) implies that the condition (A) is equivalent with

$$\mathbf{P}_x\left(\,R_t \in \overline{G}\,\right) = 1$$

for every $t \geq t_0$, but this follows from the Skorohod problem $\mathrm{SE}(B., G)$, since that says that $R_t \in \overline{G}$ almost surely for every $t \geq 0$ for every starting point $x \in \overline{G}$.

Suppose now that $f \in C_0^{1,2}([0,\infty) \times \mathbb{R}^d)$ satisfies the boundary condition $-\partial_{\nu(x)} f(t,x) \geq 0$ for every $x \in \partial G \times (0,\infty)$.

We notice that the relation (11.10) also implies that the condition (B) stating

$$\widetilde{Z}_t = f(t, \widetilde{X}_t) - \int_{t_0}^{t} [\,\widetilde{X}_u \in G\,] Af(u, \widetilde{X}_u)\,\mathrm{d}u$$

is a $(\mathbf{P}, (\mathscr{M}_t^{t_0}))$-submartingale is equivalent with

$$Z_t = f(t, R_t) - \int_{t_0}^{t} [\,R_u \in G\,] Af(u, R_u)\,\mathrm{d}u$$

being a $(\mathbf{P}_x, (\mathscr{F}_t))$-submartingale where $Af(u,x) = (\partial_u f + \frac{1}{2}\triangle f)(u,x)$. We will verify this as the last part of this proof.

So let's show that Z is a submartingale. We know that R is a continuous semimartingale since it satisfies the Skorohod equation

$$R_t = B_t - \tfrac{1}{2}\int_0^t \nu(R_s)\,\mathrm{d}L_s = B_t + K_t$$

Since the process K is of locally finite variation, $\langle R, R\rangle_t = \langle B, B\rangle_t = tI$. Therefore, we may apply Itō formula to compute $f(t, R_t)$ and we obtain that

$$\begin{aligned} f(t, R_t) &= f(t_0, R_{t_0}) + \int_{t_0}^t \partial_u f(u, R_u)\,\mathrm{d}u + \int_{t_0}^t \nabla f(u, R_u)\cdot\,\mathrm{d}R_u \\ &\quad + \tfrac{1}{2}\int_{t_0}^t \triangle f(u, R_u)\,\mathrm{d}u \\ &= f(t_0, R_{t_0}) + \int_{t_0}^t Af(u, R_u)\,\mathrm{d}u + \int_{t_0}^t \nabla f(u, R_u)\cdot\,\mathrm{d}R_u \end{aligned}$$

Since $\mathrm{d}R_u = \mathrm{d}B_u - \frac{1}{2}\nu(R_u)\,\mathrm{d}L_u$ by the Skorohod equation, we can write this identity as

$$\begin{aligned} f(t, R_t) &= f(t_0, R_{t_0}) + \int_{t_0}^t Af(u, R_u)\,\mathrm{d}u + \int_{t_0}^t \nabla f(u, R_u)\cdot\,\mathrm{d}B_u \\ &\quad - \tfrac{1}{2}\int_{t_0}^t \partial_{\nu(R_u)} f(u, R_u)\,\mathrm{d}L_u \end{aligned}$$

and since $1 = [\,R_u \in G\,] + [\,R_u \in \partial G\,]$, we obtain that

$$Z_t = f(t_0, R_{t_0}) + \int_{t_0}^t [\,R_u \in \partial G\,]Af(u, R_u)\,\mathrm{d}u + M_t + H_t$$

where M_t is a bounded continuous local martingale, which means it is a bounded martingale. The process H_t is an increasing process, since by assumption $-\partial_{\nu(x)} f(u, x) \geq 0$ for every $x \in \partial G$ and therefore, $\mathrm{d}H_t = -\frac{1}{2}\partial_{\nu(x)} f(u, x)\,\mathrm{d}L_t \geq 0$.

We are now almost done, since this implies that

$$(Z_t - Z_s)[\,C\,] = (M_t - M_s)[\,C\,] + (H_t - H_s)[\,C\,] + (H^{(2)}_{s,t})[\,C\,]$$

where $C \in \mathscr{F}_s$ and

$$H^{(2)}_{s,t} = \int_s^t [\,R_u \in \partial G\,]Af(u, R_u)\,\mathrm{d}u.$$

By Fubini's theorem, we may change the order of integration and expectation, and thus

$$\mathbf{E}_x\,[\,C\,]|H^{(2)}_{s,t}| \;\leq \mathbf{E}_x\,|H^{(2)}_{s,t}| \;\leq \|\,Af\,\|_\infty \int_s^t \mathbf{E}_x\,[\,R_u \in \partial G\,]\,\mathrm{d}u.$$

With the help of Itō formula and the semimartingale property of R we can show (Excercise) that $\mathbf{P}_x\,(\,R_t \in \partial G\,) = 0$ and therefore,

$$\mathbf{E}_x\,[\,C\,]H^{(2)}_{s,t} \;= 0.$$

Since H is an increasing process, we also have that

$$\mathbf{E}_x\,[\,C\,](H_t - H_s) \;\geq 0.$$

Since M is a martingale, we have that $\mathbf{E}_x\,[\,C\,](M_t - M_s) = 0$. Therefore,

$$\mathbf{E}_x\,(Z_t - Z_s)[\,C\,] = \mathbf{E}_x\Big((M_t - M_s) + (H_t - H_s) + (H^{(2)}_{s,t})\Big)[\,C\,] \geq 0$$

which implies that Z is a submartingale since the integrability and that Z_t is $\mathscr{F}_t$-measurable are straight forward to check. □

11.11. **Corollary.** *When G has a C^2-boundary, then reflecting Brownian motion R exists, is pathwise unique and is a strong Markov process.*

11.3. **RBM from transition probability density.** There is another way to define reflecting Brownian motion. Now that we know that reflecting Brownian motion is Markov process, we can also construct it from its transition probability density.

In one dimensional case we note that the transition function is given by

$$P_t f(x) = \mathbf{E}_x\, f(R_t) = \mathbf{E}_x\, f(|B_t|) = \frac{1}{\sqrt{2\pi t}} \int_0^\infty f(y)(e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x+y)^2}{2t}})\,\mathrm{d}y$$

since we know the transition probability density of the Brownian motion and thus, we can compute the transition function of reflecting Brownian motion by change of integration variable. From this we can read that the transition probability density p of reflecting Brownian motion is

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}}(e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x+y)^2}{2t}})$$

for every $t > 0$ and $x, y \geq 0$. In order to generalise this form the one dimensional domain $G = (0, \infty)$ to a domain G in $\mathbb{R}^d$, we notice that p satisfies a partial differential equation (the heat equation)

$$\partial_t p(t,x,y) = \tfrac{1}{2}\partial_x^2 p(t,x,y)$$

for $t, x, y > 0$. Moreover, at the boundary ∂G of G, which in this case is the singleton point $\partial G = \{0\}$ the partial derivative $\partial_x p$ of the transition probability density satisfies

$$\partial_x p(t,0,y) = 0$$

for every $t > 0$ and every $y \geq 0$. Finally, the transition probability density satisfies the Feller property, namely

$$\lim_{t\downarrow 0} P_t f(x) = f(x)$$

for every $f \in C_b(0,\infty)$.

We can generalize this *initial-boundary value problem* for the *heat equation* to other domains as well. The ∂_x^2 becomes the Laplacian and the derivative

$\partial_x p$ at $x = 0$ becomes the *normal derivative* at the *boundary* ∂G. Therefore, we will consider the following initial-boundary value problem

$$\begin{cases} \partial_t p(t,x,y) = \frac{1}{2}\triangle_x p(t,x,y) & \text{for } t > 0, x, y \in G \\ \partial_{\nu(x)} p(t,x,y) = 0 & \text{for } t > 0, y \in \overline{G} \text{ and } x \in \partial G, \\ \lim_{t \downarrow 0} \int_{\overline{G}} f(y) p(t,x,y)\,\mathrm{d}y = f(x) & \text{for } x \in \overline{G}. \end{cases} \tag{11.12}$$

Here $\nu(x)$ is the *outer unit normal of the boundary* ∂G at the point $x \in \partial G$.

There are many ways to proceed to reflecting Brownian motion from the equation (11.12). We could use every technique we have from analysis for partial differential equations to show that the this equation has a unique solution and verify that the solution is a transition probability density for some Markov process X i.e. verify that

1. the equation (11.12) has a unique continuous solution p,
2. the solution p is positive, i.e. $p(t,x,y) \geq 0$ for every $t > 0$, $x, y \in \overline{G}$,
3. the solution p defines a probability measure i.e.

$$\int_{\overline{G}} p(t,x,y)\,\mathrm{d}y = 1$$

4. the solution p satisfies the Chapman–Kolmogorov equation

$$p(t+s,x,y) = \int_{\overline{G}} p(t,x,z) p(s,z,y)\,\mathrm{d}z$$

5. furthermore, the solution p is symmetric in x and y i.e. $p(t,x,y) = p(t,y,x)$ for every $x, y \in \overline{G}$

All these together can be used to show that p generates a Markov process X which can furthemore shown to be a Feller process with continuous paths almost surely. Moreover, we could start from contructing the transition function directly by considering a related initial-boundary value problem

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\triangle_x u(t,x) & \text{for } t > 0, x \in G \\ \partial_{\nu(x)} u(t,x) = 0 & \text{for } t > 0, \text{ and } x \in \partial G, \\ u(0,x) = f(x) & \text{for } x \in \overline{G}. \end{cases} \tag{11.13}$$

The previous problem is reduces to this if we could replace the function f by a Dirac measure δ_y.

Since we already have constructed a reflecting Brownian motion (at least for domains with C^2-boundary), we can also proceed by starting form the reflecting Brownian motion R and verifying that its probability density function satisfies the equation (11.12) and has the properties listed above.

11.14. **Theorem.** *Suppose X is a Markov process with p as its transition probability density. Then $\widetilde{\mathbf{P}}(A) := \mathbf{P}_x\,(\,X_\cdot \in A\,)$ solves the Submartingale Problem* sMP$(I, 0, -\nu, 0, G)$ *for every $x \in \overline{G}$.*

Proof. Again we start by noting that

$$\mathbf{P}_x\,(\,X_\cdot \in A\,) = \widetilde{\mathbf{P}}(\widetilde{X}_\cdot \in A). \tag{11.15}$$

The relation (11.15) implies that the condition (A) is equivalent with

$$\mathbf{P}_x\left(\,X_t \in \overline{G}\,\right) = 1$$

for every $t \geq t_0$. This on the other hand is equivalent with

$$\int_{\overline{G}} p(t,x,y)\,\mathrm{d}y = 1$$

which is the property 3..

The condition (B) is seen to be equivalent with

$$Z_t^{(f,t_0)} = f(t,X_t) - f(t_0,X_{t_0}) - \int_{t_0}^{t} [\,X_u \in G\,] Af(u,X_u)\,\mathrm{d}u$$

being a $(\mathbf{P}_x, (\mathscr{F}_t))$-submartingale where $Af(u,x) = (\partial_u f + \frac{1}{2}\triangle f)(u,x)$ and f is smooth enough function with $\partial_{\nu(x)} f(x) \leq 0$ for every $x \in \partial G$. Now

$$\begin{aligned}\mathbf{E}_x\Big(Z_t^{(f,t_0)}\,|\,\mathscr{F}_s\Big) &= \mathbf{E}_x(f(t,X_t)\,|\,\mathscr{F}_s)\; - f(t_0,X_{t_0}) - \int_{t_0}^{s} [\,X_u \in G\,] Af(u,X_u)\,\mathrm{d}u \\ &\quad - \int_s^t \mathbf{E}_x([\,X_u \in G\,] Af(u,X_u)\,|\,\mathscr{F}_s)\;\mathrm{d}u\end{aligned}$$

Since we assumed that X is a Markov process (which follows from properties 2. − 4..) we have that

$$\mathbf{E}_x(f(t,X_t)\,|\,\mathscr{F}_s)\; = \mathbf{E}_{X_s}\,f(t,X_{t-s})\; = \mathbf{E}_{X_s}\,f(s+(t-s),X_{t-s})$$

and

$$\mathbf{E}_x([\,X_u \in G\,] Af(u,X_u)\,|\,\mathscr{F}_s)\; = \mathbf{E}_{X_s}\,[\,X_{u-s} \in G\,] Af(u,X_{u-s})$$

If we denote $g(u,x) = f(u+s,x)$ for every u, then

$$\mathbf{E}_x(f(t,X_t)\,|\,\mathscr{F}_s)\; = \mathbf{E}_{X_s}\,g(t-s,X_{t-s})$$

and since we have $Ag(u,x) = Af(u+t,x)$ we also obtain that

$$\begin{aligned}\int_s^t \mathbf{E}_x([\,X_u \in G\,] Af(u,X_u)\,|\,\mathscr{F}_s)\;\mathrm{d}u &= \int_0^{t-s} \mathbf{E}_{X_s}\,[\,X_u \in G\,] Af(u+s,X_u)\;\mathrm{d}u \\ &= \mathbf{E}_{X_s} \int_0^{t-s} [\,X_u \in G\,] Ag(u,X_u)\,\mathrm{d}u\end{aligned}$$

Moreover, since $\mathbf{E}_{X_s}\, g\,(0, X_0) = g(0, X_s) = f(s, X_s)$, we can write

$$\begin{aligned}\mathbf{E}_x\Big(Z_t^{(f,t_0)} \mid \mathscr{F}_s\Big) &= Z_s^{(f,t_0)} - f(s,X_s) + \mathbf{E}_{X_s}\, g(t-s, X_{t-s}) \\ &\quad - \mathbf{E}_{X_s} \int_0^{t-s} [\, X_u \in G\,] A g(u, X_u)\, \mathrm{d}u \\ &= Z_s^{(f,t_0)} + \mathbf{E}_{X_s}\, Z_{\,t-s}^{\,(g,0)}\end{aligned}$$

and therefore, $Z^{(f,t_0)}$ is a submartingale, if $\mathbf{E}_{X_s}\, Z_{\,t-s}^{\,(g,0)} \geq 0$ for every $t_0 < s < t$. For this it is sufficient to show that $\mathbf{E}_x\, Z^{(g,0)}{}_t \geq 0$ holds for every $t > 0$.

The expectation of the whole expression is

$$\mathbf{E}_x\, Z^{(g,0)}{}_t = \mathbf{E}_x\, g(t, X_t) \; - \mathbf{E}_x\, g(0, X_0) \; - \int_0^t \mathbf{E}_x\, A g(u, X_u) \; \mathrm{d}u = I - J.$$

The first term I becomes

$$\begin{aligned}I &= \mathbf{E}_x\, g(t, X_t) \; - \mathbf{E}_x\, g(0, X_0) \; = \int_{\overline{G}} g(t,y) p(t,x,y)\, \mathrm{d}y - \int_{\overline{G}} g(0,y) p(0,x,y)\, \mathrm{d}y \\ &= \int_{\overline{G}} (\rho_{x,y}(t) - \rho_{x,y}(0))\, \mathrm{d}y = \int_{\overline{G}} \int_0^t \rho_{x,y}'(u)\, \mathrm{d}u\, \mathrm{d}y\end{aligned}$$

where $\rho_{x,y}(u) = p(u,x,y) g(u,y)$ and therefore,

$$\rho_{x,y}'(u) = g(u,y)\partial_u p(u,x,y) + p(u,x,y)\partial_u g(u,y).$$

Since $p(u,x,y) = p(u,y,x)$ by 5. we have that $\partial_p p(u,x,y) = \partial_u p(u,y,x)$ and since p satisfies the heat equation, we also have $\frac{1}{2}\partial_x p(u,x,y) = \frac{1}{2}\partial_y p(u,x,y)$ and we may write

$$\rho_{x,y}'(u) = \tfrac{1}{2} g(u,y) \triangle_y p(u,y,x) + p(u,x,y)\partial_u g(u,y).$$

The second term J can be written as

$$\begin{aligned}J &= \int_0^t \mathbf{E}_x\, A g(u, X_u) \; \mathrm{d}u = \int_0^t \int_{\overline{G}} A g(u,y) p(u,x,y)\, \mathrm{d}y\, \mathrm{d}u \\ &= \int_0^t \int_{\overline{G}} (\partial_u g(u,y) + \tfrac{1}{2}\triangle_y g(u,y)) p(u,x,y)\, \mathrm{d}y\, \mathrm{d}u \\ &= \int_0^t \int_{\overline{G}} (\partial_u g(u,y) + \tfrac{1}{2}\triangle_y g(u,y)) p(u,y,x)\, \mathrm{d}y\, \mathrm{d}u\end{aligned}$$

If we write $p_{(x,t)}(y) = p(t,y,x)$ and $g_t(y) = g(t,y)$ we may combine the identities for I and J and we get that

$$I - J = \tfrac{1}{2} \int_0^t \int_{\overline{G}} (g_u(y) \triangle p_{(u,x)}(y) - p_{(u,x)} \triangle g_u(y))\, \mathrm{d}y\, \mathrm{d}u$$

Since

$$\begin{aligned}\nabla \cdot (g_u \nabla p_{(u,x)} - p_{(u,x)} \nabla g_u) &= g_u \triangle p_{(u,x)} + \nabla g_u \cdot \nabla p_{(u,x)} \\ &\quad - \nabla p_{(u,x)} \cdot \nabla g_u - p_{(u,x)} \cdot \triangle g_u \\ &= g_u \triangle p_{(u,x)} - p_{(u,x)} \cdot \triangle g_u\end{aligned}$$

we may use the Gauß divergence theorem[47] with $F = g_u \nabla p_{(u,x)} - p_{(u,x)} \nabla g_u$ and we obtain

$$I - J = \tfrac{1}{2} \int_0^t \int_{\partial G} g_u(y) \partial_{\nu(y)} p(u, y, x) - p(u, y, x) \partial_{\nu(y)} g(u, y)) \sigma(\,\mathrm{d}y) \,\mathrm{d}u$$

This difference is non-negative, since by equation (11.12) the $\partial_{\nu(y)} p(u, y, x) = 0$ for every $y \in \partial G$. Hence

$$I - J = \tfrac{1}{2} \int_0^t \int_{\partial G} (-p(u, y, x) \partial_{\nu(y)} g(u, y)) \sigma(\,\mathrm{d}y) \,\mathrm{d}u$$

Moreover, we kwow by 1. that $p(u, y, x) \geq 0$ for every $x, y \in \overline{G}$ and $u > 0$ and since $\partial_{\nu(y)} f(u, y) \leq 0$ for every $u > 0$ and every $y \in \partial G$ we deduce that $-p(y, y, x) \partial_{\nu(y)} g(u, y) \geq 0$ for every $u \in (0, t)$ and $y \in \partial G$ and so $I - J \geq 0$.

This was enough to show that the $Z^{(f,t_0)}$ is a submartingale and the claim follows. □

[47]Gauß divergence theorem states that

$$\int_{\overline{G}} \nabla \cdot F(x) \,\mathrm{d}x = \int_{\partial G} \nu(x) \cdot F(x) \sigma(\,\mathrm{d}x)$$