10. Skorohod's equation and reflecting Brownian motion in 1D

Now that we have shown that $R = |B|$ is a good candidate for the reflecting Brownian motion (i.e. it behaves like a Brownian motion outside 0 and it reflects towards the normal direction) we can define that

10.1. **Definition.** A process R with the same law as $|B|$ (i.e. with the same marginal distributions) is called the reflecting Brownian motion reflecting at 0.

We have seen that via Tanaka's formula that

$$R_t = |B_t| = R_0 + \beta_t + L_t$$

where $\mathrm{d}\beta_t = \operatorname{sgn}(B_t)\,\mathrm{d}B_t$ is a Brownian motion and L is a local time, which is increasing process and it only increases when $B_t = 0$. However, this also implies L only increases when $R_t = |B_t| = 0$, so in a sense we don't need the original Brownian motion B directly at all.

In 1961 A.V. Skorohod introduced his equation that shows that this really is the case, i.e. once we are given β we can in a unique way find R and L such that $R \geq 0$ and L is increasing and increases only when $R_t = 0$.

10.2. **Theorem** (Skorohod's equation)**.** *Suppose $y\colon [0,\infty) \to \mathbb{R}$ is a continuous function and $y(0) \geq 0$. Then there are unique pair of functions (z, a) such that*

1. *$z(t) = y(t) + a(t)$ for every t*
2. *$z(t) \geq 0$ for every t*
3. *a is increasing, continuous, $a(0) = 0$ and for every t*

$$\int_0^t [\, z(s) \neq 0 \,]\,\mathrm{d}a(s) = 0$$

Moreover, the function a is given by a formula

$$a(t) = \sup_{0 \leq s \leq t} y(s)^- \tag{10.3}$$

Before proving this, let's verify that once we are given β, then R and L satisfy the three conditions. The first part is Tanaka's formula. The second also holds since $R = |B|$. The third one follows from Theorem 9.6, since it says that for every t

$$\begin{aligned} 0 = \int_0^t |B_s|\,\mathrm{d}L_s = \int_0^t R_s\,\mathrm{d}L_s = \int_0^t [\, R_s \neq 0\,] R_s\,\mathrm{d}L_s \\ \geq \int_0^t [\, R_s \geq \alpha\,] R_s\,\mathrm{d}L_s \geq \alpha \int_0^t [\, R_s \geq \alpha \,]\,\mathrm{d}L_s \end{aligned}$$

for every $\alpha > 0$ almost surely.This implies that for every $\alpha_1, \alpha_2 > 0$

$$0 = \int_0^t [\, \alpha_1 \leq R_s < \alpha_2 \,]\,\mathrm{d}L_s$$

and since $[\, x > 0 \,] = \sum_n [\, (n+1)^{-1} \leq x < n^{-1} \,]$ we can deduce that

$$\int_0^t [\, R_s \neq 0 \,] \, \mathrm{d}L_s = \int_0^t [\, R_s > 0 \,] \, \mathrm{d}L_s = 0$$

almost surely. This also implies that if $R_0 = 0$, then

$$L_t = \sup_{0 \leq s \leq t} (-\beta_s) > 0$$

for every $t > 0$. Let's now prove the Skorohod's equation.

Proof. **The existence**. To show the existence it is enough to verify that if a is given by the formula (10.3) and $z := y + a$, then (z, a) satisfy the properties 1. – 3.. The property 1. is trivially valid, since we define z so that it satisfies 1.

The 2. property is similarly simple, since

$$z(t) = y(t) + a(t) = y(t) + \sup_{s \leq t} y(s)^-$$

If $y(t) \geq 0$, then since $a(t) \geq 0$ we have $z(t) \geq 0$. So we may assume that $y(t) < 0$. Then $-y(t)^- = y(t)$ and so

$$y(t) + \sup_{s \leq t} y(s)^- = -y(t)^- + \sup_{s \leq t} y(s)^- \geq -y(t)^- + y(t)^- = 0$$

The 3. property has many parts. Since y is continuous, we see that a is continuous. By definition $a(0) = 0$ and also as a supremum over larger and larger set, it is increasing.

Let's prove the last part and for this let's show first that

$$\int_0^t [\, z(s) > \varepsilon \,] \, \mathrm{d}a(s) = 0.$$

Since z is continuous function as a sum of two continuous functions, the set $\{\, s \geq 0 : \ z(s) > \varepsilon \,\}$ is an open subset of $\mathbb{R}$. Therefore, it is a countable union of open intervals (s_n, t_n). This means that

$$\int_0^t [\, z(s) > \varepsilon \,] \, \mathrm{d}a(s) = \sum_{n=1}^{\infty} (a(t_n) - a(s_n)).$$

On such an interval we have

$$\forall s \in (s_n, t_n] : y(s) = z(s) - a(s) > \varepsilon - a(s) \geq \varepsilon - a(t_n)$$

since a is increasing. This implies that

$$\forall s \in (s_n, t_n] : y(s)^- \leq (\varepsilon - a(t_n))^-$$

If $a(t_n) \leq \varepsilon$ then $y(s)^- = 0$ for every $s \in (s_n, t_n]$ and

$$a(t_n) = \sup_{s \leq t_n} y(s)^- = \sup_{s \leq s_n} y(s)^- = a(s_n).$$

If $a(t_n) > \varepsilon$, then $(\varepsilon - a(t_n))^- = a(t_n) - \varepsilon < a(t_n)$ and so

$$a(t_n) = \sup_{s \le t_n} y(s)^- = \max(a(s_n), a(t_n) - \varepsilon) = a(s_n)$$

Thus, we always have $a(t_n) = a(s_n)$ and the integral vanishes. Since this holds for every $\varepsilon > 0$, we obtain that claim in the same way as we did when we verified that the local time of Brownian motion and the reflecting Brownian motion satisfies the same condition.

The uniqueness. Suppose (z, a) and (z', a') are two solutions to the Skorohod equation. This means that $b := z - z' = a - a'$, since $z - a = y = z' - a'$. Now since a and a' are increasing functions by 3., the function b is of finite variation and we may integrate with respect to it. In particular, we know that (from the Itō formula for instance as a special case) that

$$\int_0^t b(s)\,\mathrm{d}b(s) = \tfrac{1}{2}(b(t)^2 - b(0)^2)$$

Since $a(0) = a'(0) = 0$, we also have $b(0) = 0$ and therefore,

$$\int_0^t b(s)\,\mathrm{d}b(s) = \tfrac{1}{2}b(t)^2 \ge 0$$

On the other hand

$$\begin{aligned}\int_0^t b(s)\,\mathrm{d}b(s) = \int_0^t (z(s) - z'(s))\,\mathrm{d}b(s) = \int_0^t z(s)\,\mathrm{d}a(s)\\ + \int_0^t z'(s)\,\mathrm{d}a'(s) - \int_0^t z'(s)\,\mathrm{d}a(s) - \int_0^t z(s)\,\mathrm{d}a'(s)\end{aligned}$$

Since a is increasing by 3. and $z \ge 0$ by 2. we can estimate

$$0 \le \int_0^t z(s)\,\mathrm{d}a(s) = \int_0^t [\, z(s) > 0\,] z(s)\,\mathrm{d}a(s) = 0$$

where the last equality follows by the same approximation procedure for the set $[\, z(s) > 0\,]$ as above. Therefore,

$$\int_0^t z(s)\,\mathrm{d}a(s) + \int_0^t z'(s)\,\mathrm{d}a'(s) = 0$$

so we are have shown that

$$0 \le \int_0^t b(s)\,\mathrm{d}b(s) = -\int_0^t z'(s)\,\mathrm{d}a(s) - \int_0^t z(s)\,\mathrm{d}a'(s) \le 0$$

since z, z' are both positive (by 2.) and a and a' are both increasing (by 3.) Therefore,

$$0 = \int_0^t b(s)\,\mathrm{d}b(s) = b(t)^2 = (z(t) - z'(t))^2 = (a(t) - a'(t))^2$$

and the uniqueness follows. □