0. Introduction

0.1. **Randomness.** On this course we will be studying stochastic processes that live on (bounded) domains in plane or higher dimensional Euclidean spaces. Here a domain mean the open and connected set.

The name of the course included many parts and first we will briefly go through the meaning of the words.

We will interpret the word *stochastic* to mean *random that can be approached via probability* and the word *process* to mean a phenomenon that changes in time or is *time dependent.*

Therefore, we will think that stochastic processes are random phenomena that change in time and can be treated with probabilities.[1] We will be studying few simple random models. These are simplifications of the interesting phenomena that appear in nature, financial markets or gambling, for instance.

Let's start briefly by analysing the connection between certain phenomenon and its mathematical and random model (but only on an conceptual level). Heat equation is (one) familiar mathematical model for the heat conduction coming from physics. This model shows that the temperature distribution in a body tries to flatten in such a way that the heat flows from the hotter spots to colder spots.[2] Mathematically the heat equation is a second order linear *partial differential equation* $\partial_t u = \partial^2_{x_1} u + \dots \partial^2_{x_n} u = \triangle u$ and the predictions that the model provides are obtained by analysing this equation and the behavior of its solutions.

Closer inspection reveals that the temperature is a consequence of the heat motion of the smallest building blocks (particles) of the body and this motion is very irregular in the lattice formed by these tiny particles. The hotter the place more these particles are moving. The heat conduction now follows since the particles collide with their neighbors and the faster particles will then be losing more of their kinetic energy in these collision that those that move slower. In average sense, this model corresponds to the particles differential equation above.

In practise, the movement of these tiny particles is best modelled as purely random motion. This means that we can think that the model of heat conduction is stochastic model and this is one example when this approach is practical.

[1]This interpretation is general, but the "time" might not correspond to an intuitive picture we usually have

[2]or the coldness flows from the colder spots to the hotter spots, what ever.

However, this is the last time we really consider any modelling questions or even connections with the real world problems but instead we will concentrate on simple stochastic models. We will first start with *Brownian motion* and analyse some of its properties. We will generalise some of these properties to bit more complicated processes by applying *stochastic differential equations* which we will interpret to mean equations that depend in a relatively simple way from the Brownian motion. In fact, in some sense the main concept of the course is Brownian motion[3] and Brownian motion is in sense exactly the microscoping idealisation of the heat motion of the particles in correspondence with the heat equation.

0.2. **Brownian motion and boundary value problems.** Brownian motion "lives", however, in an unbounded domain namely in the whole underlying space. This means that Brownian motion can at least in principle reach every place in the space given enough time. Since we want to study processes that are Brownian motion like, but who "live" in more restricted domains, we need methods for "restricting" the Brownian motion.

One of techniques is called stopping[4]. In this case, we create a process that stops immediately when the process living in the whole space would have left the given domain.

This stopping technique is in a close relation with so called *Dirichlet boundary value problems.* The first example of this is from Shizuo Kakutani from 1944. He considered the Laplace equation and its Dirichlet boundary value problem in the domain D. This problem is: *find such a function* u *in* $C^2(D)$, that satisfies

$$\begin{cases} \triangle u(x) = 0, & x \in D \subset \mathbb{R}^2 \\ u(y_n) \to f(x), & x \in \partial D. \end{cases}$$

for every *suitable sequence* $(y_n) \subset D$ that $y_n \to x$ and for given continuous function $f\colon \partial D \to \mathbb{R}$. He showed that this problem has a solution in very general domains D and that this solution is also unique. But, what is more, he showed that the solution can be represented with the help of Brownian motion, namely

$$u(x) = \mathbf{E}_x\, f\left(B_\tau\right)$$

where τ is the *the first exit time* of Brownian motion from the domain D.

We will soon define Brownian motion and also quite soon we will prove the result of Kakutani by using the generalisation of the integration by Kiyoshi Itô.

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[3]or Brownian motion type process

[4]instead of stopping we also generally use the term *killing*. The vocabulary of probability theory is quite dark. . .

With the same Itô's integration theory we can define the *stochastic differential equations*. With the help of stochastic differential equations, we will generalise Kakutani's representation to a larger class of partial differential equations. Namely, we will consider the following problem *find such a function u in $C^2(D)$*, that satisfies

$$\begin{cases} a(x) : D^2u(x) + b(x)\nabla u(x) = 0, & x \in D \subset \mathbb{R}^d \\ u(y_n) \to f(x), & x \in \partial D. \end{cases}$$

for every[5] *suitable sequence* $(y_n) \subset D$ that $y_n \to x$ and for given continuous function $f \colon \partial D \to \mathbb{R}$. We will show that this has also a unique solution in rather general domains D and it can be given with the help of Brownian motion. In other words

$$u(x) = \mathbf{E}_x\, f\,(X_\tau)$$

wher τ is the *life time* of the stochastic process X in the domain D and where the process X is the *solution* of the stochastic differential equation

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t$$

as long as $X_t \in D$[6] and $\sigma\sigma^\top = 2a$.

0.3. **Reflecting Brownian motion and the Neumann problem.** The first method is not the only way to make sure that the stochastic process does not escape from the given domain. Another way, which we will be studying as well, is the method of *reflecting*. The intuitive picture is that when the process "hits" the boundary of the domain D, it is reflected back inside. How this is actually done is highly nonunique, so we will only consider one case: the reflection towards the normal direction.

The simplest example is reflection in half space, i.e. when $D = \mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \mathbb{R}_+$. If B is the Brownian motion and it starts moving from a point x in a half space D we can intuitively define *the reflecting Brownian motion* by

$$R_t := (B_t^{(1)}, \ldots, B_t^{(d-1)}, |B_t^{(d)}|), \quad t \geq 0.$$

As we will find out later, this can be written as a differential equation with the so called Itô–Tanaka formula as

$$\mathrm{d}R_t := \mathrm{d}B_t + \mathrm{e}_d\,\mathrm{d}L_t^B = \mathrm{d}B_t + \tfrac{1}{2}\mathrm{e}_d\,\mathrm{d}L_t^R$$

where L^X is the so called *local time of the process X* on the boundary of the half space. This stochastic differential equation is called as *Skorohod's equation.*

[5]With the notation $a(x) : D^2u(x)$ we mean the function $\sum_{j,k=1}^d a_{jk}(x)\partial_{jk}u(x)$. We will return to these notations later.

[6]this means that τ is the moment, when X_t "stops" being a solution

This equation can be generalised to other domains as well and one way to define (as we will later show) the local time is by computing the following limit [7]

$$L_t = \lim_{\varepsilon\downarrow 0} \varepsilon^{-1} \int_0^t [\,\mathrm{dist}(\partial D, X_t) \le \varepsilon\,]\,\mathrm{d}s.$$

In 1976 Gunnar–Arnvid Brosamler showed that the reflecting Brownian motion has a counterpart of the Kakutani's representation. We will consider the following Neumann boundary value problem for the Laplace equation: *find such a function u in* $C^2(D)$, that satisfies

$$\begin{cases} \triangle u(x) = 0, & x \in D \subset \mathbb{R}^2 \\ \partial_\nu u(x) = f(x), & x \in \partial D. \end{cases}$$

for given continuous function f that has vanishing integral over the boundary. Note, that the domain needs to be regular enough so that we can somehow define the normal derivative on the boundary. Brosamler showed that for C^3 domains[8] the solution (upto a constant) can be represented with the help of reflecting Brownian motion and local time by the following formula

$$u(x) = \lim_{t\to\infty} \tfrac{1}{2}\mathbf{E}_x \int_0^t f(R_t)\,\mathrm{d}L_t^R\,.$$

All the examples we have had the processes have been *Markov processes* that are continuous. Itô integration theory gives a nice addition to these by providing us a lot of *martingales* that have really nice properties.

When we try to understand this Brosamler's representation and its generalisations (by for instance replacing the Laplace operator by a more general operator) and when we try to understand the "fine structure" of the boundary in more detail, it turns out that we can take advantage of a bit more general view point that can be easily managed with the Itô integration theory. This leads us to consider the connection between Markov processes and operator semigroups that Masatoshi Fukushima introduces in 1967 and finalised in 1971. This way we end up with quite regular Markov processes that are called *Hunt processes*[9].

Putting it simply, the *transition operator semigroup* relates every suitable function f in the domain D and every time instance $t > 0$ a new function with the rule:

$$T_t f(x) := \mathbf{E}_x\, f(X_t)$$

[7] I will come back to this notation

[8] this regularity means that in small enough neighborhoods of the boundary, the boudnary can be described as a graph of a C^3-function $x \in \partial D \cap U$ iff $0 = \varphi(x)$ and $\varphi \in C^3(U)$.

[9] The name Hunt comes from Gilbert Agnew Hunt who was first to introduce them

and we just set $T_0 f = f$. This family of operators $(T_t)_{t\geq 0}$ is called a semigroup, since the Markov property (as we will later demonstrate) implies that $T_{t+s}f = T_tT_sf$ or in other words, if the compostion of functions $\circ$ is treated as the operation, $(T_t, \circ)$ is closed, associative and contains the neutral element T_0.

The transition operators connect in an natural way the Hunt processes with abstract Cauchy problem and we can use this connection in a fruitful way for analysing the properties of these processes. We will also give a general definition for the local time via so called *Revuz correspondence* and we will use *Fukushima decomposition* to help with the Itô integration theory.