Department of Mathematics and Statistics
Stochastic processes on domains
Excercise problem sheet 6
(To be returned by Tuesday 05.05.2015)

Note. In the Problems 1-12 the j,k and n are always integers.
Note. The Problems 1-4 provide a proof of the Lévy's Characterisation Theorem.

1. Suppose M is a 1-dimensional continuous local $(\mathscr{F}_t)$-martingale. Show that for every $\lambda \in \mathbb{C}$, the process

$$Z_t^\lambda := \exp\Big(\lambda M_t - \tfrac{1}{2}\lambda^2 \langle M, M \rangle_t\Big)$$

is a local martingale. (Hint. Itō with f such that $f(M, \langle M, M \rangle) = Z^\lambda$).

2. Suppose $X = (X^1, \ldots, X^d)$ is a d-dimensional continuous local $(\mathscr{F}_t)$-martingale, suppose $X_0 = 0$ and suppose $\langle X^j, X^k \rangle_t = [\, j = k\,]t$. Show that for every $f = (f_1, \ldots, f_d)$ with $f_j \in L^2(\mathbb{R}^+)$ the process

$$Y_t^f := \exp\Big(i \sum_{k=1}^d \int_0^t f_k(X_s)\,\mathrm{d}X_s^k + \tfrac{1}{2} \sum_{k=1}^d \int_0^t f_k^2(X_s)\,\mathrm{d}s\Big)$$

is a complex and bounded martingale. (Hint. Problem 1 with suitable λ and M).

3. Suppose X is a $(\mathscr{F}_t)$-adapted continuous d-dimensional process, $X_0 = 0$ and let Y^f be the process as in Problem 2. Suppose for every $f = (f_1, \ldots, f_d)$ with $f_k \in L^2(\mathbb{R}^+)$ the process Y^f is complex and bounded $(\mathscr{F}_t)$-martingale. Show that

$$\mathbf{E}_0\,[\,A\,]\exp\Big(i\xi \cdot (X_t - X_s)\Big) = \mathbf{P}_0\,(\,A\,)\exp(-\tfrac{1}{2}|\xi|^2(t-s))$$

holds for every $s < t < u$, every $\xi \in \mathbb{R}^d$ and every $A \in \mathscr{F}_s$. (Hint. $f = (f_1, \ldots, f_d)$ with $f_k(s) = \xi_k[\, s \le u\,]$)

4. Assume the same as in Problem 3. Show that for every $s < t$ the increment $X_t - X_s$ is independent from $\mathscr{F}_s$ and show X has the same expectation and variance as Brownian motion (i.e. show that X is $(\mathscr{F}_t)$-Brownian motion).

In Problem 5-7 we look at convex functions.

5. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a convex function. Show that if $x' < x$ are two points on interval $(-r/2, r/2)$, then

$$f(x') - f(x) \leq |x - x'| \frac{f(-r) - f(x')}{r + x}.$$

(Hint: in this case $-r < x' < x$ and then convexity)

6. Continuing with Problem 5. show that if f is convex and $|f| \leq C$ on the interval $(-r, r)$ then

$$|f(x) - f(x')| \leq \frac{4C}{r} |x - x'|$$

for every $x, x' \in (-r/2, r/2)$ or in other words, f is locally Lipschitz if f is locally bounded. (Hint: estimate the fraction in Problem 5, and repeat the construction for $x > x'$.)

7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex function. Let $\psi \in C^\infty$ which is zero outside an interval (a, b) for $a < b < 0$, which is positive and which integrates to 1. Define

$$f_n(x) = n \int_{-\infty}^{\infty} f(x + y) \psi(ny) \, \mathrm{d}y.$$

Show that f_n is convex. (Hint. use the definition of convexity directly).

8. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex function and let ψ and f_n be as in Problem 7. Show that $f_n'(x) \to f_-'(x)$.

9. Suppose $t \mapsto p(t, x, y) \in C^1(\mathbb{R}_+)$ and $(x, y) \mapsto p(t, x, y) \in C^2(D) \cap C^1(\overline{D})$ and that p satisfies the heat equation

$$\begin{cases} \partial_t p(t, x, y) = \frac{1}{2} \triangle_x p(t, x, y), & \forall (t, x, y) \in (0, \infty) \times D \times D \\ \partial_\nu p(t, x, y) = 0 & \forall (t, x, y) \in (0, \infty) \times \partial D \times \overline{D} \\ p(0, x, \cdot) = \delta_x & \forall x \in \overline{D} \end{cases}$$

where $\partial_\nu p = \nu(x) \cdot \nabla_x p(t, x, y)$. Show that if p is unique, then

$$p(t + s, x, y) = \int_{\overline{D}} p(t, x, z) p(s, z, y) \, \mathrm{d}z$$

holds for every $t, s > 0$ and $x, y \in D$. (Hint. differentiate both sides with s and use the uniqueness of p)

10. Suppose p is as in Problem 9. Show that

$$\int_{\overline{D}} p(t, x, y) \, \mathrm{d}x = 1$$

for every $t \geq 0$ and $x \in \overline{D}$.

11. Suppose p is as in Problem 9 and assume that p is unique. If we know in addition that $p(t, x, y) \geq 0$ for $t \geq 0$, show that p is a probability transition density of some Feller process (X_t) with the state space $\overline{D}$.