Department of Mathematics and Statistics
Stochastic processes on domains
Excercise problem sheet 5
(To be returned by Wednesday 22.04.2015)

Note. In the Problems 1-12 the j,k and n are always integers.

1. Suppose M and N are bounded martingales and M and N are independent. Show that

$$\langle M, N \rangle = 0.$$

2. Suppose f is a continuous function on the boundary of a ball $D_r(x)$ and η is the first exist time from the ball $D_r(x)$. Show that

$$\mathbf{E}_x f(B_\eta) = \int f(y)\mu(\,\mathrm{d}y)$$

where μ is the normalised surface measure of the sphere $\partial D_r(x)$. (Hint: if f is an indicator function, show that the μ has to rotation invariant. You may assume that you know that then it must be the surface measure)

3. Suppose $w(x) = \mathbf{E}_x w(B_\eta)$ as in the proof of Lemma 7.5. Show that

$$w(x) = \int_{D_r(x)} w(y)\varphi(|y-x|)\,\mathrm{d}y$$

for every $\varphi\colon \mathbb{R} \to \mathbb{R}^+$ such that $\int \varphi(t)\,\mathrm{d}t = 1$ and $\varphi(t) = 0$ outside interval $(r/2, r)$. (Hint. use Problem 2. and Fubini to the right-hand side to separate w and φ.)

4. Suppose $w(x) = \mathbf{E}_x w(B_\eta)$ as in the proof of Lemma 7.5. Show that w is $C^\infty(G)$. (Hint. use previous problem 3. and differentiate. You may assume the existence of C^∞ functions that vanish outside $(r/2, r)$.

5. Show that the for every $z \in G$

$$\mathbf{P}_z\,(\,\tau \le t\,) = \lim_{n\to\infty} \mathbf{E}_z\,\mathbf{P}_{B(n^{-1})}\left(\tau \le t - n^{-1}\right)$$

and that $z \mapsto \mathbf{E}_z \, \mathbf{P}_{B(n^{-1})} \left(\tau \leq t - n^{-1} \right)$ is continuous (even C^∞) for every n. (Hint. Markov property and the transition probability density.)

6. Show that the for every $x \in \partial G$ and every $(x_n) \subset G$ such that $x_n \to x$ it holds that

$$\mathbf{P}_x \left(\tau \leq t \right) \leq \liminf_{n \to \infty} \mathbf{P}_{x_n} \left(\tau \leq t \right)$$

(Hint: use Problem 5 to deduce this lower semicontinuity property by approximating from below by continuous functions)

7. Show that if x is a regular point on the boundary and $(x_n) \subset G$ such that $x_n \to x$, then

$$\mathbf{P}_{x_n} \left(\tau \leq t \right) = 1$$

for every $t > 0$. (Hint. Problem 6.)

8. Show that 0 is a regular point of $(0, 1)$ for 1-dimensional Brownian motion without using flat cone condition. (Hint. Blumenthal 0-1 -law).

9. Prove the Blumenthal's 0-1 -law. i.e. show that when $\mathscr{F}_0$ is augmented history of Brownian motion, then if $A \in \mathscr{F}_{0+} = \mathscr{F}_0$, we either have $\mathbf{P}_x \left(A \right) = 0$ or $\mathbf{P}_x \left(A \right) = 1$. (Hint. consider the random variable $[\,A\,][\,A\,]$ and use Markov property to deduce that $\mathbf{E}_x \left[\,A\,\right]\left[\,A\,\right] = \mathbf{P}_x \left(A \right)^2$.)

10. Suppose $\frac{1}{2}\triangle u = g$ in domain G. If u is $C^2(G)$ and g is bounded, show that

$$Z_t = u(B_t) - \int_0^t g(B_s) \, \mathrm{d}s$$

is a continuous local martingale in $[0, \tau)$ for every starting point x.

11. Suppose $\frac{1}{2}\triangle u = qu$ in domain G. If u is $C^2(G)$ and $q \leq 0$, show that

$$Z_t = u(B_t) e^{-\int_0^t q(B_s) \, \mathrm{d}s}$$

is a continuous local martingale in $[0, \tau)$ for every starting point x.

12. Suppose $\frac{1}{2}\triangle u = g$ in domain G and $u = f$ on ∂G. If u is $C^2(G)$ and it is continuous in $\overline{G}$, and f is bounded, show that

$$u(x) = \mathbf{E}_x \, f(B_\tau) \; - \mathbf{E}_x \int_0^\tau g(B_s) \, \mathrm{d}s$$