Department of Mathematics and Statistics
Stochastic processes on domains
Excercise problem sheet 4
(To be returned by Monday 30.03.2015)

Note. In the Problems 1-12 the j,k and n are always integers.

1. Suppose M and N are bounded martingales. Show that

$$\langle M, N \rangle^{\tau} = \langle M^{\tau}, N \rangle.$$

(Hint: use uniqueness in Theorem 6.10. and Optional Stopping Theorem.)

2. Let N and M be continuous local martingales. Show that

$$\langle M, N \rangle = \langle N, M \rangle$$

(Hint: use Theorem 6.10 and uniqueness)

3. Let N_1, N_2 and M be continuous local martingales and $\alpha \in \mathbb{R}$. Show that

$$\langle M, N_1 + \alpha N_2 \rangle = \langle M, N_1 \rangle + \alpha \langle M, N_2 \rangle$$

(Hint: use Theorem 6.10 and uniqueness)

Note: In the following two Problems, you may assume that it is known that

$$K_1 \cdot (K_2 \cdot H) = K_1 K_2 \cdot H$$

for locally bounded processes K_1, K_2 and $H \in \mathscr{A}$.

4. Let K and H be locally bounded processes (Definition 6.18) and let N be continuous local martingale. Let $Y = H \cdot M$ be a continuous local martingale. Show that

$$K \cdot Y = KH \cdot M$$

(Hint: take N a continuous local martingale and use Theorem 6.20 twice to express $\langle K \cdot Y, N \rangle$ as an stochastic integral with respect to a process $\langle M, N \rangle$, and then use uniqueness in Theorem 6.20).

5. Let K and H be locally bounded processes and M and N be continuous local martingales. Show that

$$\langle K \cdot M, H \cdot N \rangle = KH \cdot \langle M, N \rangle.$$

6. Let $X = \alpha K \cdot B$, where B is 1-dimensional Brownian motion and $K_t = B_t^2$. Verify that

$$\langle X, X \rangle_t = \alpha^2 \int_0^t B_s^4 \, \mathrm{d}s$$

(Hint: use Problem 5 and the fact that $t = \langle B, B \rangle_t$).

7. Determine continuous local martingale M and locally finite variation process $A \in \mathscr{A}$, such that $A_0 = M_0 = 0$ and

$$B_t^4 = B_0^4 + M_t + A_t$$

where B is 1-dimensional Brownian motion. (Hint: $B_t^4 = f(B_t)$ when $f(x) = x^4$ and Itō's formula.)

8. Use Itō's formula to find a polynomial function $f(x, y)$ such that

$$B_t^6 - f(B_t, t)$$

is a continuous local martingale. (Hint: Let $f_0(x, y) = x^6 + c_1 x^4 t + c_2 x^2 t^2 + c_3 t^3$. With Itō obtain a equations for the coefficients c_1, c_2, c_3)

Note. We say that a process X is in class (DL), if $\{ X^\tau : \tau$ is a *bounded* stopping time $\}$ is uniformly integrable.

9. Let M be a local martingale. Show that if M is in class (DL), then it is a martingale. (Hint: Let (τ_n) is the sequence as in the Definition 5.10. and let τ be a bounded stopping time. Show that $X_{\tau_n \wedge \tau} \to X_\tau$ almost surely, that $\mathbf{E}_x X_{\tau_n \wedge \tau} = \mathbf{E}_x X_0$ and that $\{X_{\tau_n \wedge \tau}\}_n$ is uniformly integrable. Then have a look at Lemma 5.2. in lecture notes and the Problem 4 in the Excercise sheet 3.)

10. Let $X_t = |B_t|^{-1}$ where B_t is *three-dimensional* Brownian motion that starts from $x \neq 0$. In lectures we showed that X_t is a local martingale. Show that it is not in

class (DL). (Hint: show directly that it is *not a martingale* by considering the mean $\mathbf{E}\, X_t$ of X_t.)

11. We know that for every $x \neq 0$ in the plane (i.e. in $\mathbb{R}^2$) that

$$\mathbf{P}_x \,(\, \tau_0 = \infty \,) = 1$$

where x is the starting point of *two-dimensional* Brownian motion. Show that this holds also for $x = 0$. (Hint: let $\nu = \tau_r$ be the first hitting time to the sphere of radius r and use strong Markov property at τ. Use this to deduce the claim.)

12. Suppose there exists a 1-dimensional continuous semimartingale X such that

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t$$

where a and b are bounded and $C^2(\mathbb{R}, \mathbb{R})$-functions. Let $Z = f(X_t)$ for $f \in C^2(\mathbb{R}, \mathbb{R})$. Use Itō's formula to find a continuous local martingale M and process $A \in \mathscr{A}$ such that $A_0 = M_0 = 0$ and

$$Z_t = Z_0 + M_t + A_t$$

Furthermore, compute $\langle\, M, M \,\rangle$ (Hint. Use the formula in Problem 5 for the computation).