Department of Mathematics and Statistics
Stochastic processes on domains
Excercise problem sheet 3
(To be returned by Friday 13.03.2015)

Note. In the Problems 1-12 the j,k and n are always integers.

1. Suppose τ_1 and τ_2 are $(\mathscr{F}_{t+})$-stopping times. Show that $\mathscr{F}_{\tau_1^+} \subset \mathscr{F}_{\tau_2^+}$.

2. Let X be a positive and integrable random variable. Show that the family $\{ Y : \ |Y| \ \leq X \}$ is uniformly integrable. (Hint: estimate $|Y|[Y > m]$ from above by random variable depending only on X and m).

3. Assume that $\mathbf{E}\, X_n^2 \leq M$ for every n. Show that $\{ X_n : \ n \in \mathbb{N} \}$ is uniformly integrable. (Hint: $|X_n|m \leq X_n^2$ when $|X_n| \geq m$.)

4. Assume $X_n \to X$ in almost surely and $\{ X_n : \ n \in \mathbb{N} \}$ is uniformly integrable. Show that $X_n \to X$ in L^1-sense. (Hint: let $\phi_n(x) = x[\,|x| \leq n\,] + n[\,x > n\,] - n[\,x < -n\,]$ and write $X_k - X = (X_k - \phi_n(X_k)) - (X - \phi_n(X)) + (\phi_n(X_k) - \phi_n(X)) = I_1 + I_2 + I_3$ and estimate term I_j each separetely).

5. Let $X_t = B_{t \wedge s}$ be a stopped 1-dimensional Brownian motion at time instance $s \in (0, \infty)$ (i.e. at constant stopping time). Show that $X_t = \mathbf{E}\,(X_\infty \,|\, \mathscr{F}_t)$ for every $t \in (0, \infty)$ and deduce that X is uniformly integrable martingale. (Hint: X_t^- and $-X_t^+$ are supermartingales if X is a martingale, see Lemma 5.2. and Theorem 5.3.)

6. Define

$$p_0(t, x, y) = \frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\right)$$

for every $t > 0$, and $x, y \in \mathbb{R}$ and $x, y \geq 0$. Verify that

$$\partial_t p_0(t, x, y) = \tfrac{1}{2}\partial_y^2 p_0(t, x, y) \quad \text{and} \quad p_0(t, 0, y) = 0$$

for every $t > 0$ and $x, y > 0$. This is the transition probability density function of 1-dimensional killed BM which is killed at zero.

Note. In Problems 7, 8 and 9 the $(\mathscr{F}_t)$ is a given filtration and (X_t) is a right continuous $(\mathscr{F}_t)$-supermartingale.

7. Let τ be a simple stopping time, say $\tau \in \{t_1, \ldots, t_n\}$. Show that X_τ is $\mathscr{F}_\tau$-measurable and $\mathscr{F}_\tau = \mathscr{F}_{\tau^+}$.

8. Let $s \le t < u$ and τ a stopping time. Show that

$$\mathbf{E}\left([\tau > t](X_u - X_t) \,|\, \mathscr{F}_s\right) \le 0$$

(Hint: You can use the fact that $\mathbf{E}\left(Z \,|\, \mathscr{H}\right) = \mathbf{E}\left(\mathbf{E}\left(Z \,|\, \mathscr{G}\right) \,|\, \mathscr{H}\right)$ for every σ-algebras $\mathscr{H} \subset \mathscr{G}$ and then you can apply the supermartingale property).

9. Let τ and be a simple stopping time, say $\tau \in \{t_1, \ldots, t_n\}$. Let us assume that $s = t_0 \le t_1 < t_2 < \cdots < t_n$. Show that and

$$\mathbf{E}\left(X_\tau \,|\, \mathscr{F}_s\right) \le X_s.$$

(Hint: show

$$X_\tau - X_s = \sum_{k=0}^{n-1} [\tau > t_k](X_{t_{k+1}} - X_{t_k})$$

and use this identity with Problem 8.)

10. Let $X_t = e^{sB_t - s^2 t/2}$. Show that it is a martingale with respect to the history of Brownian motion. Hint: you are on right track if you have arrived to

$$\mathbf{E}\left(X_u \,|\, \mathscr{F}_t\right) = e^{sB_t} e^{-s^2 u/2} \mathbf{E}\, e^{sB_{u-t}}$$

11. Let $\tau_a = \inf\{\ t > 0 :\ B_t = a\ \}$ be the first hitting time of *1-dimensional* Brownian motion to the point a. Let $a < x < b$ and $X_t = B_t^{\tau_a \wedge \tau_b}$. Show that X_t is a bounded martingale for every $x \in (a, b)$.

12. Assume the same as in 11. Show that

$$a\mathbf{P}_x\left(\tau_a < \tau_b\right) + b\mathbf{P}_x\left(\tau_b < \tau_a\right) = x$$

and show that

$$\mathbf{P}_x\left(\tau_a < \tau_b\right) = \frac{b - x}{b - a} = 1 - \mathbf{P}_x\left(\tau_b > \tau_a\right).$$

(Hint: Optional Stopping Theorem for a bounded martingale. For the latter explain first why $\mathbf{P}_x\left(\tau_a = \tau_b\right) = 0$ and then you have two equations for the probabilities).